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# The quasi-periodic Hamiltonian Hopf bifurcation

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## Abstract

We consider the quasi-periodic dynamics of non-integrable perturbations of a family of integrable Hamiltonian systems with normally  $1 : -1$  resonant invariant tori. In particular, we focus on the supercritical quasi-periodic Hamiltonian Hopf bifurcation and address the persistence problem of the singular foliation into invariant quasi-periodic tori near the bifurcation point. With the help of KAM theory, we show that the singular torus foliation survives a small perturbation and that the persisting tori in this foliation form Cantor families. A leading example is the Lagrange top near gyroscopic stabilization weakly coupled with a quasi-periodic oscillator.

Mathematics Subject Classification: 37G99, 37J20, 37J40, 70E17

(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

The centre-saddle bifurcation and the Hamiltonian Hopf bifurcation are the only ways in which an equilibrium can lose its stability in a generic one-parameter family of Hamiltonian systems. While the equilibrium ceases to exist altogether after a centre-saddle bifurcation, the persistence of the equilibrium is not in question during a Hamiltonian Hopf bifurcation, cf [51], where the linear behaviour changes from elliptic to hyperbolic. A periodic orbit may furthermore lose its stability in a (Hamiltonian) period doubling bifurcation, see [55].

Corresponding to these three bifurcations there are several Diophantine conditions that have to be imposed to prove the persistence of normally elliptic invariant tori in Hamiltonian systems, cf [19, 20, 34, 44, 47, 53, 54, 56, 63, 67]. The condition

$$|\langle \omega, k \rangle| \geq \gamma |k|^{-1} \quad \text{for all } k \in \mathbb{Z}^m \setminus \{0\} \quad (1)$$

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bounds the internal frequencies  $\omega_1, \dots, \omega_m$  away from resonance and is also imposed to prove the persistence of normally hyperbolic invariant tori, cf [19, 20, 38, 44, 67]. In the periodic case  $m = 1$  it expresses that the period is finite.

The remaining conditions also involve the normal frequencies  $\omega_j^N$ . The first Mel'nikov condition

$$|\langle \omega, k \rangle + \omega_j^N| \geq \gamma(1 + |k|)^{-1} \quad \text{for all } k \in \mathbb{Z}^m \quad (2)$$

ensures in particular that the normal frequencies do not vanish. Where this condition is violated a quasi-periodic centre-saddle bifurcation may occur, cf [40]. In the periodic case  $m = 1$  the first Mel'nikov condition expresses that the Floquet multipliers  $\exp(\omega_j^N/\omega_1)$  are bounded away from 1. Then the implicit function theorem can be used to prove the persistence of the periodic orbit. Correspondingly, the approach in [8, 9, 74] yields the persistence of lower dimensional tori already under (1) together with (2). The price to pay is that one loses control on the normal linear behaviour.

The second Mel'nikov condition excludes normal internal resonances of (normal) order two and thus effectively allows us to control the normal linear behaviour. Let us write this condition in two parts, as

$$|\langle \omega, k \rangle + 2\omega_j^N| \geq \gamma(1 + |k|)^{-1} \quad \text{for all } k \in \mathbb{Z}^m \quad (3)$$

and

$$|\langle \omega, k \rangle + \omega_j^N + \omega_l^N| \geq \gamma(1 + |k|)^{-1} \quad \text{for all } k \in \mathbb{Z}^m. \quad (4)$$

Let us furthermore assume that the first Mel'nikov condition (2) is satisfied, otherwise (3) is violated as well. In the periodic case  $m = 1$  the condition (3) then expresses that the Floquet multipliers  $\exp(\omega_j^N/\omega_1)$  are bounded away from  $-1$ . Correspondingly, the (normally) elliptic tori may undergo a frequency halving bifurcation where (3) is violated, cf [15].

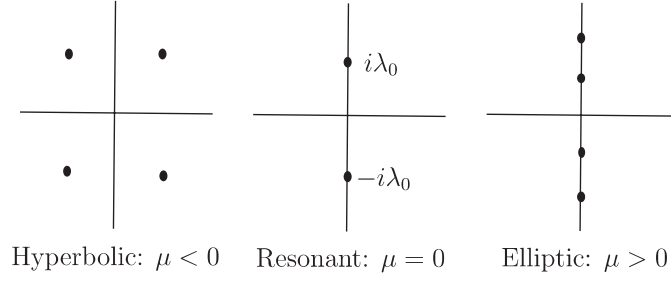
The present paper is concerned with condition (4). We follow [74] and choose Floquet coordinates that turn a given resonance violating (4) into  $\omega_j^N + \omega_l^N = 0$ . As shown in [49, 75] the perturbed tori can still be reduced to the Floquet form, but both (normally) elliptic and hyperbolic tori may occur. The result proven in [16] states that generically both types of invariant  $m$ -tori do occur and that furthermore the family of  $m$ -tori undergoes a linear Hamiltonian Hopf bifurcation. When passing through the  $1 : -1$  resonance  $\omega_j^N + \omega_l^N = 0$  the normal linear behaviour changes from elliptic to hyperbolic.

Under generic conditions on the higher order terms, a nonlinear Hamiltonian Hopf bifurcation takes place. Next to the  $m$ -tori this also involves families of  $(m + 1)$ -tori and  $(m + 2)$ -tori. This yields a stratification of the product of phase space and parameter space. The resonances excluded by (1)–(4) lead to a Cantorization of this stratification. To fix thoughts we assume from now on that the two normal frequencies in the  $1 : -1$  resonance are the only normal frequencies, whence  $(m + 2)$ -tori are maximal tori of dimension  $m + 2 = d$  equal to the number of degrees of freedom.

**Remark 1.1.** Usually the second Mel'nikov condition is formulated in such a way that it includes the Diophantine condition

$$|\langle \omega, k \rangle + \omega_j^N - \omega_l^N| \geq \gamma(1 + |k|)^{-1} \quad \text{for all } k \in \mathbb{Z}^m.$$

As shown in [45] and also following from [49, 75], elliptic  $m$ -tori that violate this condition nevertheless persist as elliptic invariant tori, i.e. under preservation of their normal linear behaviour.



**Figure 1.** Eigenvalue configuration of the linearized Hamiltonian as  $\mu$  passes through 0.

### 1.1. Hamiltonian Hopf bifurcation

We consider the quasi-periodic dynamics in Hamiltonian systems with a normally resonant invariant torus. If  $d$  is the number of the degrees of freedom, we assume that  $d \geq 4$ . In particular, we consider the normal  $1 : -1$  resonance, which involves a quasi-periodic Hamiltonian Hopf bifurcation. This bifurcation is a natural extension of the Hamiltonian Hopf bifurcation of equilibria (in two-degrees-of-freedom systems) [51]. For completeness we briefly recall the latter.

**1.1.1. Hamiltonian Hopf bifurcation for equilibria.** Let  $H = H(z; \mu)$  be a 1-parameter family of Hamiltonian functions on  $\mathbb{R}^4 = \{z_1, z_2, z_3, z_4\}$  with the standard symplectic 2-form  $\sigma = dz_1 \wedge dz_3 + dz_2 \wedge dz_4$ . For each  $\mu$ , the Hamiltonian  $H(z; \mu)$  has  $z = 0$  as an equilibrium point. Moreover,

- the Hamiltonian  $H$  is in generic (or non-semisimple)  $1 : -1$  resonance at  $\mu = 0$ , that is, the quadratic part of  $H$  has the normal form

$$\hat{H}_2(z; 0) = \lambda_0(z_1 z_4 - z_2 z_3) + \frac{1}{2}(z_3^2 + z_4^2),$$

where  $\lambda_0 \neq 0$ . Notice that  $N = \frac{1}{2}(z_3^2 + z_4^2)$  is the nilpotent part of  $\hat{H}_2(z; 0)$ , while  $\lambda_0 S = \lambda_0(z_1 z_4 - z_2 z_3)$  is the semisimple part;

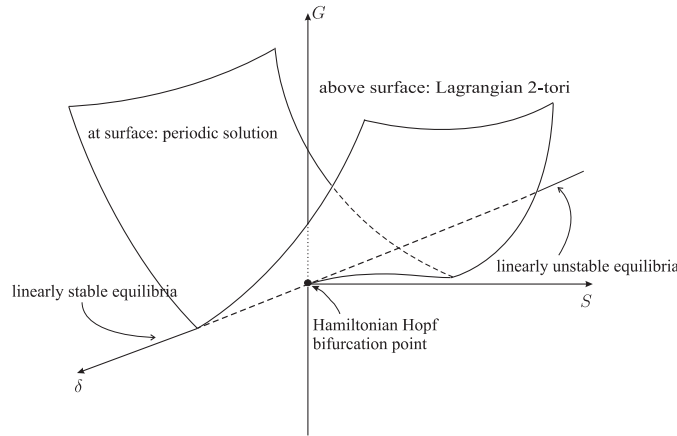
- as  $\mu$  passes through the critical value  $\mu = 0$ , the equilibrium  $z = 0$  changes from linearly unstable to linearly stable, that is, the linearized Hamiltonian vector field associated with  $\hat{H}$  (at  $z = 0$ ) changes from hyperbolic into elliptic, see figure 1.

Then, the Hamiltonian  $H$  has the Birkhoff normal form

$$\hat{H} = (v_1(\mu) + \lambda_0)S + N + v_2(\mu)M + \frac{1}{2}b(\mu)M^2 + c_1(\mu)SM + c_2(\mu)S^2 + \dots, \quad (5)$$

where  $M = \frac{1}{2}(z_1^2 + z_2^2)$  and the dots denote the higher order terms as functions of  $S, M$  and  $\mu$  [51]. Note that a (truncated) normal form  $\hat{H}$  is invariant under the  $\mathbb{S}^1$ -action generated by the flow of the semisimple quadratic part  $S$ ; normalization consists of pushing this symmetry through the Taylor series up to an arbitrary order [51]. Then,  $\hat{H}$  is Liouville-integrable with the integrals  $S$  and  $\hat{H}$ . We say that the family  $\hat{H}$  undergoes a *Hamiltonian Hopf bifurcation* at  $\mu = 0$ , if  $(\partial v_2 / \partial \mu)(0) \neq 0$  and  $b(0) \neq 0$ .

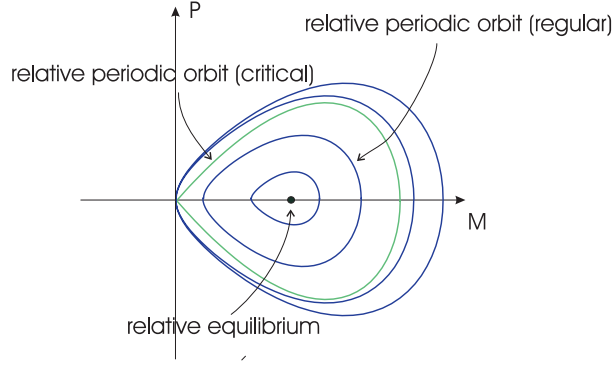
The inverse images of the energy-momentum map  $EM := (S, \hat{H}) : \mathbb{R}^4 \rightarrow \mathbb{R}^2$  fibre the phase space into invariant sets of the Hamiltonian system  $X_{\hat{H}}$  associated with  $\hat{H}$ : for a regular value  $m = (s, h) \in \mathbb{R}^2$ , the set  $EM^{-1}(m) \subset \mathbb{R}^4$  is an  $X_{\hat{H}}$ -invariant 2-torus; for a critical value  $m$ , this set is an invariant ‘pinched’ 2-torus, a circle or an equilibrium. In this way the



**Figure 2.** Local stratification by invariant tori near the Hamiltonian Hopf bifurcation point  $(G, S, \delta) = (0, 0, 0)$  in the parameter space. The surface is a piece of the swallowtail catastrophe set [64].

phase space becomes a *ramified torus bundle* or *singular foliation*, where the regular fibres are  $X_{\hat{H}}$ -invariant tori. This gives rise to a stratification in the product of the phase space and the parameter space: the  $(\mu, s, h)$ -space is split into different strata according to the dimension of the tori  $EM^{-1}(s, h)$ . In the supercritical case where  $b(0) > 0$ , it is shown in [51] that the energy-momentum map  $(\hat{H}, S) : \mathbb{R}^4 \rightarrow \mathbb{R}^2$  is (locally left-right) equivalent to the map  $(G, S)$  with  $G = N + \delta(\mu)M + M^2$ . Thus, the inverse images of  $(G, S)$  provide a local foliation of the phase space near the equilibrium. Associated with this foliation we obtain a local stratification in the  $(\delta, S, G)$ -space given by a piece of the swallowtail catastrophe set: the stratum of equilibria at the crease and the thread, the stratum of periodic solutions at the smooth part of the surface and the stratum of Lagrangian 2-tori at the open region above the surface (excluding the thread). For a sketch see figure 2. The question is what remains of this singular foliation when the family  $\hat{H}$  is slightly perturbed to  $\tilde{H} = H$ , where  $H$  is the original Hamiltonian. By an application of the inverse function theorem, the perturbed family  $\tilde{H}$  has a normalized quadratic part of the same form as  $\hat{H}_2$ , leading to the same normal form as (5), compare with [16, section 2]. Now the same local (singularity theoretical) analysis on the normal form  $\tilde{H}$  applies to the perturbation  $\tilde{H}$ . From this we conclude that the singular foliation is preserved if  $\tilde{H}$  is  $\mathbb{S}^1$ -symmetric. Expressing the dynamics in the variables  $N, M$  and  $P$  effectively reduces the  $\mathbb{S}^1$ -symmetry; see figure 3 for the projection of the resulting reduced flow to the  $(M, P)$ -plane. In the case of nearly integrable (i.e. not necessarily  $\mathbb{S}^1$ -symmetric) perturbation it is shown in [51] that the geometrical picture in figure 2 still remains the same, except for the Lagrangian 2-tori. Application of KAM theory [2, 46, 56] yields the persistence for most of these. In fact, the persistent 2-tori form a Whitney-smooth Cantor family, due to Diophantine non-resonance conditions [23, 62].

**1.1.2. Quasi-periodic Hamiltonian Hopf bifurcation.** The local stratification by tori in a parameter space as sketched in figure 2 can also be found in families of integrable Hamiltonian systems of more degrees of freedom with an invariant normally  $1 : -1$  resonant torus, compare with figure 4(a). Then, we may ask the same persistence question as before, now regarding the (higher dimensional) invariant tori associated with this stratification. In view of KAM theory, we expect Whitney-smooth Cantor families of invariant tori in the foliation to survive small



**Figure 3.** Integral curves of reduction of  $\hat{H}$  with  $S = 0$ , projected on the  $(M, P)$ -plane, where  $P = z_1 z_3 + z_2 z_4$ . Relative equilibria correspond to periodic orbits in the full system  $\hat{H}$  and relative periodic orbits to invariant 2-tori.

perturbations, compare with figure 4(b). These families are parametrized over domains with positive Lebesgue measures of corresponding dimension.

Let us specify our present setting. Consider the phase space  $M = \mathbb{T}^m \times \mathbb{R}^m \times \mathbb{R}^4 = \{x, y, z\}$  with the symplectic 2-form

$$\sigma = \sum_{i=1}^m dx_i \wedge dy_i + \sum_{j=1}^2 dz_j \wedge dz_{2+j},$$

where  $\mathbb{T}^m = \mathbb{R}^m / (2\pi\mathbb{Z}^m)$ . Notice that  $M$  admits the free  $\mathbb{T}^m$ -action

$$(\theta, (x, y, z)) \in \mathbb{T}^m \times M \mapsto (\theta + x, y, z) \in M.$$

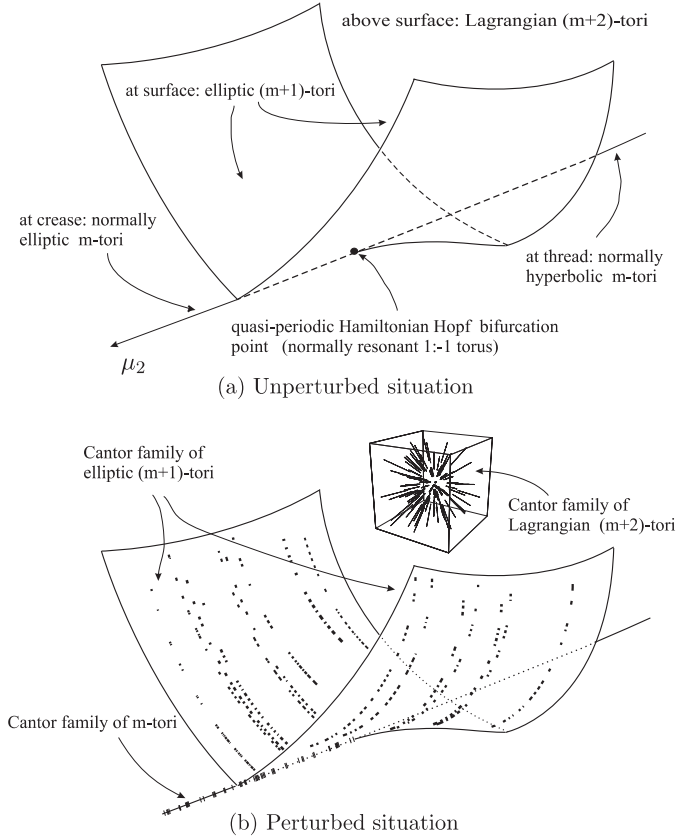
A Hamiltonian function is said to be  $\mathbb{T}^m$ -symmetric if it is invariant under this  $\mathbb{T}^m$ -action. We consider the family of  $\mathbb{T}^m$ -symmetric Hamiltonian functions

$$H(x, y, z; \nu) = \langle \omega(\nu), y \rangle + \frac{1}{2} \langle Jz, \Omega(\nu)z \rangle + \dots,$$

where  $\nu \in \mathbb{R}^p$  and where the dots denote higher order terms. Notice that the union  $T = \bigcup_{\nu} T_{\nu}$ , where  $T_{\nu} = \{(x, y, z; \nu) : (y, z) = (0, 0)\}$ , is a  $p$ -parameter family of invariant  $m$ -tori of  $H$ . We assume that the torus  $T_{\nu_0}$  is in *generic normal 1 : -1 resonance*, that is, the Floquet matrix  $\Omega_0 = \Omega(\nu_0)$  has a double pair of purely imaginary eigenvalues with a non-trivial nilpotent<sup>5</sup> part, compare with figure 1. Moreover, we assume that  $H$  is invariant under the  $\mathbb{S}^1$ -action, generated by the quadratic term  $S = \frac{1}{2} \langle Jz, \Omega_0^s z \rangle$ , where  $\Omega_0^s$  is the semisimple part of  $\Omega_0$ , cf the previous section. For  $m = 0$  and  $m = 1$ , we arrive at the settings of [51] and [61], respectively. From now on we restrict to the case where  $m \geq 2$ .

Denote by  $M/\mathbb{T}^m$  the orbit space of the  $\mathbb{T}^m$ -action. For each value  $a \in \mathbb{R}^n$ , we introduce the space  $M_a = (M/\mathbb{T}^m) \cap \{y = a\}$ , identified with  $\mathbb{R}^4 = \{z_1, z_2, z_3, z_4\}$ . Then, the function  $H$  induces a two-degrees-of-freedom Hamiltonian  $H_a = H(a, z; \nu)$  on the reduced space  $M_a$  with the symplectic 2-form  $dz_1 \wedge dz_3 + dz_2 \wedge dz_4$ . The Hamiltonian family  $H = H(x, y, z; \nu)$  of  $(m+2)$ -degrees-of-freedom is said to undergo a (non-integrable) *quasi-periodic Hamiltonian Hopf bifurcation* at  $\nu = \nu_0$ , if the reduced family  $H_a = H_a(z; \nu)$  has a Hamiltonian Hopf bifurcation and the frequencies are non-degenerate. As before we restrict to the supercritical case. Then, from the discussion in the previous section, the phase space  $M$ , near the resonant torus  $T_{\nu_0}$ , is foliated by invariant  $m$ -, by  $(m+1)$ - and by  $(m+2)$ -tori of the integrable Hamiltonian  $H$ . This defines a local stratification, as sketched in figure 4(a), in a suitable

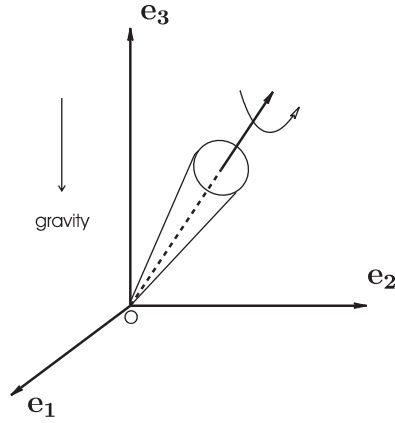
<sup>5</sup> In this case,  $\Omega_0$  is also said to be in generic or non-semisimple 1 : -1 resonance.



**Figure 4.** (a) Singular foliation by invariant tori of the unperturbed integrable Hamiltonian  $H$  near the resonant torus, in the supercritical case. (b) Sketch of Cantor families of surviving Diophantine invariant tori in the singular foliation of the perturbed, nearly integrable Hamiltonian  $\tilde{H}$ .

parameter space. Notice that the Hamiltonian Hopf bifurcation point corresponds to the normally  $1 : -1$  resonant torus  $T_{v_0}$ . Now our goal is to investigate the persistence of the local singular foliation by tori near this resonant torus, when  $H$  is perturbed to a non-integrable Hamiltonian.

The presence of the normal  $1 : -1$  resonance gives rise to multiple Floquet exponents, see figure 1. For the persistence of the  $m$ -tori near the resonance, the ‘standard’ KAM theory on the persistence of (normally) elliptic tori [19, 20, 34, 44, 47, 53, 54, 56, 63, 67] is therefore not directly applicable. Instead, we apply an adapted version of the KAM theorem [16] to the present setting. This ensures the existence of a Whitney-smooth Cantor family of surviving invariant  $m$ -tori, determined by Diophantine conditions. The persistence of the Cantor family of elliptic invariant  $(m+1)$ -tori will be obtained by a combination of a nonlinear normal form theory and an application of the ‘standard’ KAM theory [20, 44, 56, 63]. In the case of the Lagrangian  $(m+2)$ -tori, we resort to classical KAM theory [2, 46, 62]. The main results of the present paper roughly say that the singular foliation by tori near the normally resonant torus—corresponding to the local stratification in a suitable parameter space as given in figure 4—is ‘Cantorized’ under small perturbations of  $H$ . The persistent tori form Cantor families of invariant  $m$ -,  $(m+1)$ - and  $(m+2)$ -tori which are only slightly deformed from the unperturbed ones. The stratification, associated with the surviving tori, in the parameter space is sketched



**Figure 5.** The Lagrange top in the vertical gravitational force field.

in figure 4(b). Here the stratum of the  $(m + 1)$ -tori consists of density points of  $(m + 2)$ -quasi-periodicity of large  $(2m + 4)$ -dimensional Hausdorff measure. Similarly, the stratum of the  $m$ -tori consists of density points of  $(m + 1)$ -quasi-periodicity of large  $(2m + 2)$ -dimensional Hausdorff measure. Compare also with [19].

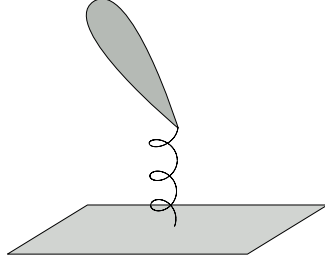
### 1.2. Perturbations of the Lagrange top

The quasi-periodic Hamiltonian Hopf bifurcation may occur when mechanical systems including the Lagrange top [24, 28, 52], the Kirchhoff top [6] and the double spherical pendulum [50] are weakly coupled with a quasi-periodic oscillator. As a leading example, we consider the Lagrange top with a quasi-periodic oscillator. The Lagrange top is an axially symmetric rigid body in a three-dimensional space, subject to a constant gravitational field such that the base point of the body-symmetry (or figure) axis is fixed in space, see figure 5. Mathematically speaking, this is a Hamiltonian system on the tangent bundle  $T SO(3)$  of the rotation group  $SO(3)$  with the symplectic 2-form  $\sigma$ . This  $\sigma$  is the pull-back of the canonical 2-form on the co-tangent bundle  $T^* SO(3)$  by the bundle isomorphism  $\tilde{\kappa} : T SO(3) \rightarrow T^* SO(3)$  induced by a non-degenerate left-invariant metric  $\kappa$  on  $SO(3)$ , where  $\tilde{\kappa}(v) = \kappa(v, \cdot)$ .

The Hamiltonian function  $H$  of the Lagrange top is obtained as the sum of the potential and kinetic energy. In the following, we identify the tangent bundle  $T SO(3)$  with the product  $M = SO(3) \times \mathfrak{so}(3)$  via the map  $v_Q \in T SO(3) \mapsto (Q, T_{\text{Id}} L_Q^{-1} v) \in SO(3) \times \mathfrak{so}(3)$ , where  $\mathfrak{so}(3) = T_{\text{Id}} SO(3)$  and  $L_Q$  denotes left-translation by  $Q \in SO(3)$ . We assume that the gravitational force points vertically downwards. Then, the Lagrange top has two rotational symmetries: rotations about the figure axis and about the vertical axis  $e_3$ . We let  $\mathcal{S} \subset SO(3)$  denote the subgroup of rotations preserving the vertical axis  $e_3$ . Then, for a suitable choice of the space coordinate system  $(e_1, e_2, e_3)$ , the two symmetries correspond to a symplectic right action  $\Phi^r$  and to a symplectic left action  $\Phi^l$  of the Lie subgroup  $\mathcal{S}$  on  $M$ . By the Noether theorem [1, 4], these Hamiltonian symmetries give rise to integrals  $\mathcal{M}^r$  and  $\mathcal{M}^l$  of the Hamiltonian  $H$ : the angular momenta along the figure axis and along the vertical axis, respectively. These integrals induce the so-called *energy-momentum map*  $\mathcal{EM} := (H, \mathcal{M}^r, \mathcal{M}^l) : M \rightarrow \mathbb{R}^3$ .

The topology of the inverse images of  $\mathcal{EM}$  provides a qualitative description of the dynamics of the top. By the Liouville–Arnold theorem [1, 4], for regular values  $(h, a, b) \in \mathbb{R}^3$ ,





**Figure 6.** The Lagrange top coupled with a vibrating table-surface by a spring.

the fibres  $\mathcal{EM}^{-1}(h, a, b)$  are invariant 3-tori, on which the flow is conditionally periodic. These Lagrangian tori are associated with three physical motions of the top: spin (rotation about the figure axis), nutation (vertical movement of the figure axis) and precession (rotation of the figure axis about the vertical axis). Since the regular values of  $\mathcal{EM}$  form an open and dense set in the image of  $\mathcal{EM}$ , the phase space is to a large extent fibred by the invariant Lagrangian 3-tori. For a critical value  $(h, a, b)$ , the fibre  $\mathcal{EM}^{-1}(h, a, b)$  forms an invariant pinched 3-torus, an invariant 2-torus or an invariant 1-torus (circle) [24, 27]. A pinched 3-torus corresponds to motions where the top tends to a vertically spinning one, as time  $t \rightarrow \pm\infty$ , while a 2-torus corresponds to a regular precession of the top (i.e. spin + precession) and a 1-torus corresponds to a vertically spinning top.

We are interested in a perturbation problem where the Lagrange top is weakly coupled with a Liouville-integrable quasi-periodic oscillator of  $n \geq 1$  frequencies. Here the coupling acts as a perturbation. For example, the base point of the top may be coupled to a vertically vibrating table-surface by a massless spring, see figure 6. In this example, the spring constant, say  $\varepsilon$ , plays the role of the perturbation parameter. More generally we consider that the (perturbed) Hamiltonian  $\mathcal{H}_\varepsilon$  has the general form

$$\mathcal{H}_\varepsilon = H + G + \varepsilon F, \quad (6)$$

where  $H$  and  $G$  are the Hamiltonians of the top and the oscillator, respectively. By Liouville-integrability, there are angle-action variables  $(\xi, \eta) \in \mathbb{T}^n \times \mathbb{R}^n$  for the Hamiltonian system of the oscillator such that the Hamiltonian  $G$  depends only on  $\eta$ . Then, the frequencies of this oscillator are given by  $\partial G / \partial \eta_j(\eta)$  for  $j = 1, \dots, n$ . The (unperturbed) Hamiltonian  $\mathcal{H}_0$  possesses invariant isotropic tori of dimension  $n + 1$ ,  $n + 2$  and  $n + 3$ . Our interest is in the fate of these invariant tori for small but non-zero  $\varepsilon$ . In particular, we address the persistence problem in the cases where the values  $a$  and  $b$  of the angular momenta  $\mathcal{M}^r$  (along the figure axis) and  $\mathcal{M}^l$  (along the vertical axis) are close to the critical values  $a = b = \pm a_0$  at which the top undergoes gyroscopic stabilization: the spinning motion becomes stabilized as the angular momentum increases. This is related to a Hamiltonian Hopf bifurcation, see section 1.1. Indeed, while the top is gyroscopically stabilized, the linear part  $\Omega_a \in \mathfrak{sp}(4, \mathbb{R})$  of the once reduced Hamiltonian system  $H_a$  (obtained by dividing out the right circle symmetry generated by  $\mathcal{M}^r$ ) changes from hyperbolic into elliptic, see figure 1.

### 1.3. Plan of the paper

The rest of the paper is organized as follows. In section 2, we review basic facts relevant for our discussion on the unperturbed Lagrange top [24]. In section 3, we introduce our perturbation problem of the Lagrange top. Section 4 is devoted to study the persistence of invariant quasi-periodic isotropic tori of dimension  $m$ ,  $m + 1$  and  $m + 2$ , respectively. Finally, in section 5,

conclusive remarks are made on the reduction of the numbers of parameters, on continua in the Cantor families of surviving tori for special perturbation models and on response solutions. Also a few open problems are briefly addressed.

## 2. The unperturbed Lagrange top

The dynamics of the unperturbed Lagrange top is extensively studied in [4, 24, 37, 48, 72]. In this section, we revisit certain known results relevant for our purposes. In particular, we consider the local dynamics of the vertically upwards spinning top when it becomes gyroscopically stabilized by a Hamiltonian Hopf bifurcation. Our discussion here closely follows [24].

### 2.1. Preliminaries

As our starting point, we consider a Hamiltonian model of the Lagrange top. Let  $\langle \cdot, \cdot \rangle$  denote the inner product of the Lie algebra  $\mathfrak{so}(3)$ , given by  $\langle A, B \rangle = -\frac{1}{2} \text{tr } AB$ , where  $\text{tr}$  stands for the trace. The tensor of inertia  $I$  of the top is a  $\langle \cdot, \cdot \rangle$ -symmetric positive linear operator on  $\mathfrak{so}(3)$ . By the axial symmetry, two of the three principal moments of inertia  $I_1, I_2$  and  $I_3$  are equal, say  $I_1 = I_2$ , implying that the map  $I$  is given by  $\text{diag}\{I_1, I_1, I_3\}$  on a suitable basis  $\{E_1, E_2, E_3\}$  of  $\mathfrak{so}(3)$ , where  $I_1, I_3 > 0$ . Later on we will assume that  $I_1 = 1$ . The Lagrange top is now described by the Hamiltonian function  $H : SO(3) \times \mathfrak{so}(3) \rightarrow \mathbb{R}$  of the form

$$H(Q, V) = \frac{1}{2} \langle I(V), V \rangle + c \langle Q E_3 Q^{-1}, E_3 \rangle, \quad (7)$$

where  $c > 0$  measures the strength of gravity and the position of the centre of mass. The Hamiltonian  $H$  has two symmetries, being the  $\mathbb{S}^1$ -actions  $\Phi^I$  and  $\Phi^r$  of the Lie group  $\mathcal{S}$  on the phase space  $SO(3) \times \mathfrak{so}(3)$ , where  $\mathcal{S} = \{P \in SO(3) : P E_3 = E_3\}$ . These two actions are given by

$$\Phi^I(P, (Q, V)) = (PQ, V)$$

and

$$\Phi^r((Q, V), P) = (QP, \text{Ad}_{P^{-1}} V).$$

By the Noether theorem [1, 4], the two rotational symmetries give rise to two integrals, namely, the angular momenta  $\mathcal{M}^r$  (along the figure axis) and  $\mathcal{M}^I$  (along the vertical axis). Following [24], one has  $\mathcal{M}^r(Q, V) = \langle I(V), E_3 \rangle$  and  $\mathcal{M}^I(Q, V) = \langle \text{Ad}_Q I(V), E_3 \rangle$ . The integrals  $H, \mathcal{M}^r$  and  $\mathcal{M}^I$  induce the energy-momentum map

$$\mathcal{EM} \equiv (H, \mathcal{M}^r, \mathcal{M}^I) : SO(3) \times \mathfrak{so}(3) \longrightarrow \mathbb{R}^3.$$

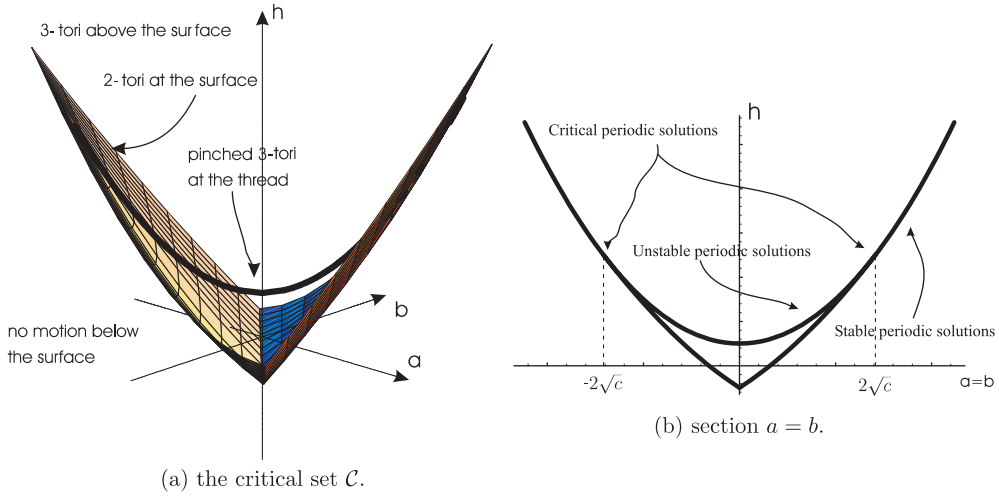
### 2.2. Reduction by symmetry

By dividing the right symmetry  $\Phi^r$  associated with the rotation about the figure axis, the original Hamiltonian system  $H$  with three degrees of freedom is reduced to a system with two degrees of freedom, see [1, 27, 65]. Indeed, for any  $a \in \mathbb{R}$ , the fibre  $(\mathcal{M}^r)^{-1}(a)$  is a  $\Phi^r$ -invariant smooth submanifold. The reduced phase space  $(\mathcal{M}^r)^{-1}(a)/\mathcal{S}$  associated with the action  $\Phi^r$  is diffeomorphic to the 4-dimensional submanifold  $\mathcal{R}_a \subset \mathbb{R}^6$  given by

$$\mathcal{R}_a = \{(u, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : u \cdot u = 1, u \cdot v = a\}, \quad (8)$$

where  $\cdot$  denotes the usual inner product on  $\mathbb{R}^3$ . The reduced Hamiltonian  $H_a : \mathcal{R}_a \rightarrow \mathbb{R}$ , induced from  $H$ , now reads

$$H_a(u, v) = \frac{1}{2I_1} (v_1^2 + v_2^2 + v_3^2) + cu_3 + \frac{1}{2} \left( \frac{1}{I_3} - \frac{1}{I_1} \right) a^2. \quad (9)$$



**Figure 7.** The set  $\mathcal{C}$  of critical values  $(h, a, b)$  of the map  $\mathcal{EM}$ . The domains with invariant tori of various dimensions of the three-degrees-of-freedom Hamiltonian  $H$ , see (7), are indicated. Unstable periodic solutions occur at the one-dimensional thread. The Hamiltonian Hopf bifurcation takes place at the two points  $(1/(2I_1) + c/(2I_3), \pm 2\sqrt{c}, \pm 2\sqrt{c})$  in the section  $a = b$ , where stable and unstable periodic solutions are separated.

The symplectic structure on  $SO(3) \times \mathfrak{so}(3)$  induces a symplectic 2-form  $\omega_a$  on  $\mathcal{R}_a$  given by

$$\omega_a(u, v)((\hat{x}, \hat{y}), (\hat{p}, \hat{q})) = x \cdot q - y \cdot p + v \cdot (v \times p), \quad (10)$$

where  $(\hat{x}, \hat{y}) = (x \times u, x \times v + y) \in T_{(u,v)}\mathcal{R}_a$  and  $(\hat{p}, \hat{q}) = (p \times u, p \times v + q) \in T_{(u,v)}\mathcal{R}_a$ , compare with [24, 65]. Here  $\times$  denotes the standard cross product of  $\mathbb{R}^3$ . The corresponding Hamiltonian vector field  $X^a$  to  $H_a$  with respect to the 2-form  $\omega_a$  is given by

$$X^a(u, v) = \left( \frac{\partial H_a}{\partial v} \times u \right) \frac{\partial}{\partial u} + \left( \frac{\partial H_a}{\partial u} \times u + \frac{\partial H_a}{\partial v} \times v \right) \frac{\partial}{\partial v}. \quad (11)$$

Note that the points  $p_{\pm}(a) = \pm(0, 0, 1, 0, 0, a) \in \mathcal{R}_a$  are two (isolated) equilibria of  $X_a$ , which are relative equilibria of the full Hamiltonian  $H$ , see (7). They correspond to a vertically spinning top. Finally, we notice that from the map  $\mathcal{EM}$ , one obtains the reduced energy-momentum map  $\mathcal{EM}_a$  of the reduced Hamiltonian system  $H_a$ , where

$$\mathcal{EM}_a : (u, v) \in \mathcal{R}_a \mapsto (H_a(u, v), v_3) \in \mathbb{R}^2. \quad (12)$$

The singular foliation into the invariant tori in the Lagrange top can be reconstructed from the fibration of  $\mathcal{EM}_a$ , see [24] for details. For a sketch of this foliation see figure 7.

### 2.3. Gyroscopic stabilization

In view of the persistence problem announced in section 1.2, we need to know the local dynamics of the Lagrange top as the vertically upwards spinning motion becomes gyroscopically stabilized. According to [24, 28], this local behaviour is characterized by a supercritical Hamiltonian Hopf bifurcation [51]. In the following, we revisit this fact and show that the Floquet matrices of the periodic solutions (associated with the rotations about the figure axis) close to gyroscopic stabilization form a universal matrix unfolding. The latter claim is of importance for an application of [16] to our persistence problem. Observe that these periodic solutions correspond to the relative equilibria  $p_+(a) = (0, 0, 1, 0, 0, a) \in \mathcal{R}_a$ , where

$\mathcal{R}_a$  is the once reduced phase space given by (8) and where the angular-momentum values  $a$  are close to a critical value  $a_0$  for which the Lagrange top undergoes gyroscopic stabilization.

From now on, for simplicity we assume that  $I_1 = 1$ ; this can always be realized by a time-scaling. Recall from section 2.1 that for the physical parameter  $c$ , we have that  $c > 0$ . Furthermore, we denote by  $P_a \in SO(3) \times \mathfrak{so}(3)$  the periodic orbit of the full Hamiltonian  $H$ , see (7), corresponding to the relative equilibria  $p_+(a)$ .

**2.3.1. Floquet matrix as a universal unfolding.** In this subsection, we show that the Floquet matrices of the periodic orbits  $P_a$ , associated with relative equilibria  $p_+(a)$ , form a universal matrix unfolding. By [24], there exist local symplectic coordinates  $z = (z_1, z_2, z_3, z_4)$  near the equilibrium  $p_+(a)$  in the reduced phase space  $\mathcal{R}_a$ , in which

- the symplectic 2-form  $\sigma_a$  of  $\mathcal{R}_a$  takes the form  $dz_1 \wedge dz_3 + dz_2 \wedge dz_4$ ;
- the once reduced Hamiltonian  $H_a$ , see (9), has the local form

$$\begin{aligned} \mathcal{H}_a = & \frac{1}{8}(a^2 - 4c)(z_1^2 + z_2^2) + \frac{1}{2}(z_3^2 + z_4^2) - \frac{1}{2}a(z_1z_4 - z_2z_3) + \frac{1}{6}(z_1z_4 - z_2z_3)^2 \\ & - \frac{1}{8}(z_1^2 + z_2^2)(z_1z_4 - z_2z_3) + \frac{1}{96}(a^2 + 8c)(z_1^2 + z_2^2)^2 + O(|z|^6), \end{aligned} \quad (13)$$

where the constant term is dropped.

By (13), the linear part of the Hamiltonian vector field corresponding to  $\mathcal{H}_a$  (with respect to the 2-form  $dz_1 \wedge dz_3 + dz_2 \wedge dz_4$ ) is given by  $\dot{z} = \Omega_{a,c}z$ , where

$$\Omega_{a,c} = \begin{pmatrix} 0 & \frac{1}{2}a & 1 & 0 \\ -\frac{1}{2}a & 0 & 0 & 1 \\ c - \frac{a^2}{4} & 0 & 0 & \frac{1}{2}a \\ 0 & c - \frac{a^2}{4} & -\frac{1}{2}a & 0 \end{pmatrix} \in \mathfrak{sp}(4, \mathbb{R}). \quad (14)$$

This  $\Omega_{a,c}$  is a Floquet matrix of the periodic orbit  $P_a$  associated with the relative equilibrium  $p_+(a)$ . In the following, we treat  $a$  and  $c$  as parameters and reparametrize to  $(\mu_1, \mu_2) = (\frac{a}{2}, \frac{1}{4}a^2 - c)$ . In these new parameters, the Floquet exponents of the periodic orbit  $P_a$ , i.e. the eigenvalues of  $\Omega_{a,c}$ , are given by

$$\lambda_{\mu_1, \mu_2} = \pm i(\mu_1 \pm \sqrt{\mu_2}). \quad (15)$$

As the parameter  $\mu_2$  passes through zero a transition of stability of periodic orbits  $P_a$  takes place, compare with figure 1. In other words, a linear Hamiltonian Hopf bifurcation [51] takes place at  $\mu_2 = 0$ . The gyroscopic stabilization occurs at  $\mu_2 = 0$ , i.e.  $|a| = 2\sqrt{c}$ . Since  $c > 0$ , the value  $|a| \approx 2\sqrt{c}$  is non-zero when the top is close to the gyroscopic stabilization.

**Lemma 2.1.** *The once reduced Hamiltonian  $H_a$  is in non-semisimple  $1 : -1$  resonance as the parameter  $\mu_2 = \frac{1}{4}a^2 - c$  passes through zero. Moreover, the family of the Floquet matrices  $\Omega = \Omega_{\mu_1, \mu_2}$  of the periodic orbits  $P_a$ , see (14), is a universal unfolding of  $\Omega_{\frac{1}{2}a_0, 0}$ , where  $a_0 = 2\sqrt{c}$ , in  $\mathfrak{sp}(4, \mathbb{R})$  in the sense of [3, 36].*

**Proof.** By (14) and (15), the Hamiltonian  $H_a$  is in  $1 : -1$  resonance, while the linear part of  $H_a$  is non-semisimple at  $\mu_2 = 0$ . This verifies the first part of the lemma. Let  $D_{(a_0/2, 0)}\Omega$  denote the differential of the map

$$\Omega : (\mu_1, \mu_2) \in \mathbb{R}^2 \mapsto \Omega_{\mu_1, \mu_2} \in \mathfrak{sp}(4, \mathbb{R})$$

at  $(\mu_1, \mu_2) = (a_0/2, 0)$ . To prove the second claim, we have to show that the image  $D_{(a_0/2, 0)}\Omega(\mathbb{R}^2)$  coincides with the kernel of the commutator  $\text{ad}\Omega_{a_0/2, 0}^T$ , where  $\Omega_{a_0/2, 0}^T$  denotes the transpose of the matrix  $\Omega_{a_0/2, 0}$ . A brief calculation shows that the subspace  $D_{(a_0/2, 0)}\Omega(\mathbb{R}^2) \subset \mathfrak{sp}(4, \mathbb{R})$  is spanned by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

which is a basis of  $\ker \text{ad}\Omega_{a_0/2, 0}^T$ . □

**2.3.2. Nonlinear Hamiltonian Hopf bifurcation.** In the previous section we saw that a linear Hamiltonian Hopf bifurcation takes place. Since the coefficient  $(a^2 + 8c)/96$  of the 4th order term  $(z_1^2 + z_2^2)^2$  is positive, the periodic solutions  $P_a$  of the full Hamiltonian  $H$  undergo supercritical Hamiltonian Hopf bifurcations at the critical values  $a_0 = \pm 2\sqrt{c}$ , see [24, 28, 42, 52]. By [51], the local geometry of the set of critical values of the reduced energy-momentum map  $\mathcal{EM}_a \equiv (H_a, m_a)$ , see (12), near the critical point  $p_+(a_0)$  is completely determined by the 4-jet of  $\mathcal{H}_a$ , see (13). Depending on the sign of the coefficient of the 4th order term  $(z_1^2 + z_2^2)^2$  in the normal form (13), there are two possible kinds of geometry. In the present supercritical case of positive coefficient, the local structure is a swallowtail without the tail, compare with figure 4. By [51, p 79], each point at the smooth part of this swallowtail surface represents an invariant 2-torus and each point above the surface an invariant 3-torus of the full Hamiltonian  $H$  on  $SO(3) \times \mathfrak{so}(3)$ . The ‘missing part’ of the swallowtail constitutes the set of critical values in the subcritical case.

**2.3.3. Local description of the Lagrange top at gyroscopic stabilization.** For later use, we give a local Hamiltonian model for motions of the (unperturbed) Lagrange top close to the gyroscopic stabilization. This is done by re-including the rotations about the figure axis (i.e. the orbits of the right symmetry  $\Phi^r$ ) into the reduced model (13). Indeed, let  $q_1$  and  $p_1$  denote the rotation angle and the angular momentum about the figure axis. By [61, theorem 1.24, p 37] the variables  $(q_1, p_1) \in \mathbb{S}^1 \times \mathbb{R}$ , together with the local coordinates  $(z_1, z_2, z_3, z_4) \in \mathbb{R}^4$ , provide local symplectic coordinates in a neighbourhood of the critical point  $p_+(a_0)$ , such that

- (i) the symplectic 2-form is given by  $dq_1 \wedge dp_1 + dz_1 \wedge dz_3 + dz_2 \wedge dz_4$ ;
- (ii) for each value  $a$ , the periodic orbit  $P_a$  is given by

$$P_a = \{(q_1, p_1, z) : p_1 = 0, z = 0\};$$

- (iii) the full Hamiltonian  $H$  has near  $P_a$  the local form

$$\begin{aligned} H_{\text{loc}} = & \left( \frac{1}{2I_3} (a + p_1)^2 + c \right) + \frac{1}{8} ((a + p_1)^2 - 4c)(z_1^2 + z_2^2) - \frac{1}{2} (a + p_1)(z_1 z_4 - z_2 z_3) \\ & + \frac{1}{2} (z_3^2 + z_4^2) + \frac{1}{6} (z_1 z_4 - z_2 z_3)^2 - \frac{1}{8} (z_1^2 + z_2^2)(z_1 z_4 - z_2 z_3) \\ & + \frac{1}{96} ((a + p_1)^2 + 8c)(z_1^2 + z_2^2)^2 + O(|z|^6). \end{aligned} \quad (16)$$

### 3. Weakly forced Lagrange top

In this section, we consider the Lagrange top (near gyroscopic stabilization) weakly coupled with a Liouville-integrable quasi-periodic oscillator of  $n \geq 1$  frequencies. Let  $(\xi, \eta) \in \mathbb{T}^n \times \mathbb{R}^n$  denote angle-action variables for the Hamiltonian system of the oscillator and  $G = G(\eta)$  be the corresponding Hamiltonian. The Hamiltonian  $\mathcal{H}_\varepsilon$  of the coupled system between the Lagrange top and the oscillator, in the local coordinates  $(q_1, p_1, z) \in \mathbb{T}^1 \times \mathbb{R} \times \mathbb{R}^4$ , has the general form

$$\mathcal{H}_\varepsilon(q_1, p_1, z, \xi, \eta) = H_{\text{loc}}(p_1, z) + G(\eta) + \varepsilon F(q_1, p_1, z, \xi, \eta, \varepsilon), \quad (17)$$

where  $H_{\text{loc}}$  is the local Hamiltonian (16) of the Lagrange top near gyroscopic stabilization. Note that the frequencies of the oscillator are given by  $\delta_j(\eta) = \partial G / \partial \eta_j(\eta)$  for  $j = 1, \dots, n$ . For  $\varepsilon = 0$ , the unperturbed system  $\mathcal{H}_0 = H_{\text{loc}} + G$  possesses  $(n+1)$ -,  $(n+2)$ - and  $(n+3)$ -dimensional invariant tori, compare with section 2.3 and with figure 4(a). The main question is how many of these invariant tori survive when  $|\varepsilon|$  is small but non-zero. We mention a few special examples of the perturbation model (17).

#### Example 3.1.

- (i) A vertical periodic oscillator ( $n = 1$ ): the base point of the Lagrange top is coupled with a vertically vibrating table-surface by a spring, see figure 6. In this case, both the right and left symmetries are preserved.
- (ii) A horizontal quasi-periodic oscillator ( $n = 2$ ): the base point of the Lagrange top is attached to a two-dimensional oscillator which consists of two linear oscillators oriented perpendicularly to each other. Such a (quasi-periodic) oscillator describes a Lissajous figure in the horizontal plane. Here the right symmetry of the Lagrange top is not affected by the oscillator, while the left symmetry is broken.
- (iii) A quasi-periodic ‘circular’ oscillator: a weak massless circular spring is placed on and around the figure axis, turning the rigid body into a gyrostat. We apply a quasi-periodic forcing on this spring to make it oscillate around the figure axis. In other words, the oscillation of the spring becomes a part of the spin motion of the top, which does not affect the mass distribution of the body. This means that such a ‘circular’ oscillator breaks the right symmetry but preserves the left one.
- (iv) A parametrically forced Lagrange top: in the above models, the Lagrange top is coupled to an external quasi-periodic oscillator. Another way to perturb is by varying for instance the physical quantity  $c$  in the unperturbed Hamiltonian system (7) during the experiment. The perturbed Hamiltonian  $H_\varepsilon$  now reads

$$H_\varepsilon = \frac{1}{2} \langle I(V), V \rangle + c(1 + \varepsilon f(t; \varepsilon)) \langle \text{Ad}_Q E_3, E_3 \rangle. \quad (18)$$

where  $f(t; \varepsilon)$  is a quasi-periodic function in  $t$ , meaning that

$$f(t; \varepsilon) = F(\delta_1 t, \dots, \delta_n t; \varepsilon),$$

where  $F : \mathbb{T}^n \times \mathbb{R} \rightarrow \mathbb{R}$  with rational independent frequencies  $\delta_1, \dots, \delta_n$ . By treating  $\xi = (\delta_1 t, \dots, \delta_n t) \in \mathbb{T}^n$  as the angle variables of a certain Hamiltonian  $G$ , we obtain a special case of the perturbation model (17). For an analysis of a parametrically forced pendulum see [4, 17, 18, 22, 59].

### 4. Quasi-periodic bifurcation

The perturbation model (17) of the Lagrange top coupled with a quasi-periodic oscillator, introduced in section 3, involves a quasi-periodic Hamiltonian Hopf bifurcation. Motivated

by this example, we want to study the quasi-periodic dynamics of a nearly integrable family of Hamiltonian systems, considered as perturbation of an integrable one that has an invariant  $1 : -1$  resonant torus. Recalling from section 1.1, we are given the symplectic manifold  $M = \mathbb{T}^m \times \mathbb{R}^m \times \mathbb{R}^4 = \{x, y, z\}$  with the standard 2-form  $\sigma = \sum_{i=1}^m dx_i \wedge dy_i + \sum_{j=1}^2 dz_j \wedge dz_{j+2}$ . The phase space  $M$  admits the free  $\mathbb{T}^m$ -action given by  $(\theta, (x, y, z)) \in \mathbb{T}^m \times M \mapsto (\theta + x, y, z) \in M$ . We consider a  $p$ -parameter family of  $\mathbb{T}^m$ -symmetric Hamiltonian functions  $H = H(x, y, z; \nu)$  in the form

$$H(x, y, z; \nu) = \langle \omega(\nu), y \rangle + \frac{1}{2} \langle Jz, \Omega(\nu)z \rangle + F(y, z, \nu), \quad (19)$$

where  $F$  denotes higher order terms and  $J$  is the standard symplectic  $4 \times 4$ -matrix, that is,

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \quad (20)$$

Observe that  $\Omega(\nu) \in \mathfrak{sp}(4, \mathbb{R})$  for all  $\nu$ . The Hamiltonian vector field  $X = X_H$  associated with  $H$  reads

$$X_\nu : \begin{cases} \dot{x} = \omega(\nu) + \frac{\partial F}{\partial y}(y, z, \nu), \\ \dot{y} = 0, \\ \dot{z} = \Omega(\nu)z + J \frac{\partial F}{\partial z}(y, z, \nu). \end{cases}$$

In vectorial shorthand notation, we write

$$X_\nu(y, z) = \left[ \omega(\nu) + \frac{\partial F}{\partial y}(y, z, \nu) \right] \frac{\partial}{\partial x} + \left[ \Omega(\nu)z + J \frac{\partial F}{\partial z}(y, z, \nu) \right] \frac{\partial}{\partial z}. \quad (21)$$

We see that the union  $T = \bigcup_\nu T_\nu$ , where  $T_\nu = \{(x, y, z, \nu) : (y, z) = (0, 0)\}$  is a  $p$ -parameter family of  $X_H$ -invariant  $m$ -tori. We recall that the torus  $T_{\nu_0}$  is in generic normal  $1 : -1$  resonance, if the matrix  $\Omega(\nu_0)$  has a double pair of purely imaginary eigenvalues with a non-trivial nilpotent part. Then, the Hamiltonian function  $H = H(y, z, \nu)$  generically undergoes a quasi-periodic Hamiltonian Hopf bifurcation at  $\nu = \nu_0$ , see section 1.1. Next we study the local quasi-periodic dynamics of  $H$  near the bifurcation point, i.e. near the resonant torus  $T_{\nu_0}$ . Denoting by  $\Omega^s(\nu_0)$  the semisimple part of  $\Omega(\nu_0)$ , we notice that the quadratic term  $S = \frac{1}{2} \langle Jz, \Omega^s(\nu_0)z \rangle$  in  $H$  generates a free  $\mathbb{S}^1$ -action  $\phi$  on the phase space  $M$ .

Ultimately, we are interested in the dynamics generated by a small perturbation of the  $\mathbb{T}^m$ -symmetric Hamiltonian  $H$ . This allows us to replace  $H$  by an approximation of  $H$  where this helps in the analysis. Now the  $z$ -dependent terms have the Birkhoff normal form (5), and hence we may assume that  $H$  is  $\phi$ -invariant.

Fixing  $a \in \mathbb{R}^n$  near the origin, the Hamiltonian  $H$  induces a Hamiltonian  $H_a = H_a(z; \nu)$  given by

$$H_a(z; \nu) = H(a, z; \nu) = \langle \omega(\nu), a \rangle + \frac{1}{2} \langle Jz, \Omega(\nu)z \rangle + \dots$$

on the reduced phase space  $M_a = (M/\mathbb{T}^m) \cap \{y = a\}$ , identified with the space  $\mathbb{R}^4 = \{z_1, z_2, z_3, z_4\}$ . By the  $\phi$ -invariance of  $H$ , the reduced Hamiltonian  $H_a$  is Liouville-integrable with the integral  $S$ , where  $S$  is considered as a function on  $M_a$ . Now by [51], there are local symplectic coordinates  $\hat{z} = (\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$  of the reduced phase space  $M_a$  near  $z = 0$ , such that the quadratic part  $(H_a)_2$  of the reduced Hamiltonian  $H_a$  takes the shape

$$(H_a)_2(\hat{z}, \nu) = (\lambda_0 + \mu_1(\nu))(\hat{z}_1 \hat{z}_4 - \hat{z}_2 \hat{z}_3) + \frac{\mu_2(\nu)}{2}(\hat{z}_1^2 + \hat{z}_2^2) + \frac{1}{2}(\hat{z}_3^2 + \hat{z}_4^2),$$



where  $(\mu_1(v), \mu_2(v)) \in \mathbb{R}^2$  are determined by the Floquet exponents of the full Hamiltonian system  $H$  at the torus  $T_v$ ; more precisely, the Floquet exponents at  $T_v$  are given by

$$\pm i(\lambda_0 + \mu_1(v) \pm \sqrt{\mu_2(v)}).$$

Since the torus  $T_{v_0}$  is normally  $1 : -1$  resonant we in particular have that  $\lambda_0 \neq 0$  and  $\mu_1(v_0) = 0 = \mu_2(v_0)$ . Following [51], the higher order terms in the normal form of  $H_a$ , with respect to  $(H_a)_2(\hat{z}, v = v_0)$ , are polynomials in  $\hat{S} = \hat{z}_1\hat{z}_4 - \hat{z}_2\hat{z}_3$  and  $\hat{M} = \hat{z}_1^2 + \hat{z}_2^2$ . Hence, at the relative equilibrium  $\hat{z} = 0$  the reduced system  $H_a$  has the following normal form:

$$H_a(\hat{z}; v) = (H_a)_2 + \frac{1}{2}b(a, v)\hat{M}^2 + c_1(a, v)\hat{S}\hat{M} + c_2(a, v)\hat{S}^2 + \dots,$$

where  $b, c_1$  and  $c_2$  are real coefficients and where the dots denote higher order terms, compare with (5). Now the (supercritical) quasi-periodic Hamiltonian Hopf bifurcation requires that  $(\partial\mu_2/\partial v)(v_0) \neq 0$  and  $b(0, v_0) > 0$ . Denote by  $EM_a$  the (reduced) energy-momentum map given by  $EM_a(\hat{z}; v) = (H_a(\hat{z}; v), \hat{S}(\hat{z}))$  of the reduced Hamiltonian  $H_a$  and by  $\mathcal{C}_v$  the set of the critical values of  $EM_a$  for each  $v$ . Then, as shown in [51], the graph  $(\mathcal{C}_v, \mu_2(v))$  locally near  $(\mu_2, \hat{S}, H_a) = (0, 0, 0)$  is a piece of the swallowtail surface without the tail, compare with figure 4(a). In this local picture the one-dimensional singular part of the surface is called the *crease*, and the isolated one-dimensional curve is referred to as the *thread*.

Now the graph  $(\mathcal{C}_v, \mu_2(v))$  provides a local stratification in the parameter space—associated with the singular foliation by tori near the normally resonant torus—as follows: the crease and the thread are associated with the elliptic and hyperbolic invariant  $m$ -tori  $T_v$  of the full Hamiltonian  $H$ , respectively; the smooth part corresponds to the elliptic invariant  $(m+1)$ -tori; the open region above the surface gives Lagrangian  $(m+2)$ -tori. Our main goal is to study the persistence of these tori when the integrable Hamiltonian  $H$  is perturbed into a nearly integrable one.

#### 4.1. Persistence of invariant resonant $m$ -tori

We examine the persistence of the  $p$ -parameter family  $T = \bigcup_v T_v$ ,

$$T_v = \{(x, y, z; v) : (y, z) = (0, 0)\} \subseteq M, \quad (22)$$

of the invariant  $m$ -tori of the  $(m+2)$ -degrees-of-freedom Hamiltonian system  $H$  given by (19). This torus family  $T$  consists of elliptic and hyperbolic tori, separated by the resonant torus  $T_{v_0}$ , compare with figure 4(a). The ‘standard’ KAM theory on the persistence of lower dimensional tori [19, 20, 34, 38, 44, 47, 56, 63, 67] yields the persistence only for subfamilies of the family  $T$ , containing elliptic or hyperbolic tori. The problem is that the resonant torus  $T_{v_0}$  gives rise to multiple Floquet exponents, compare with figure 1. To deal with this problem, we need the normal linear stability theorem [16], as an extension of the ‘standard’ KAM theory.

**4.1.1. Normal linear stability in the generic  $1 : -1$  resonant case.** We recall the KAM theory [16] for the  $p$ -parameter integrable Hamiltonian  $H = H(y, z, v)$  given by (19). The corresponding Hamiltonian vector field  $X = X_H$  to  $H$  has the form

$$X_v(y, z) = [\omega(v) + O(|y|, |z|)]\frac{\partial}{\partial x} + [\Omega(v)z + O(|y|, |z|^2)]\frac{\partial}{\partial z}, \quad (23)$$

where  $\Omega(v) \in \mathfrak{sp}(4, \mathbb{R})$  and  $v$  is in a small neighbourhood  $U$  of  $v_0 \in \mathbb{R}^p$ . To realize the persistence of the invariant  $m$ -torus family  $T = T_v$  given by (22), we need certain non-degeneracy and Diophantine conditions.



**Non-degeneracy condition.** We say that the family  $H$  of Hamiltonians given by (19) (or the corresponding family  $X = X_H$  of Hamiltonian vector fields, see (23)) is Broer–Huitema–Takens (BHT) *non-degenerate* [16, 20, 21, 68] at the invariant torus  $T_{v_0}$ , if the following holds:

- (a) the matrix  $\Omega_0 = \Omega(v_0)$  is invertible;
- (b) the map  $\omega \times \Omega : \mathbb{R}^p \rightarrow \mathbb{R}^m \times \mathfrak{sp}(4, \mathbb{R})$  is transversal to the submanifold  $\{\omega(v_0)\} \times O(\Omega_0)$  at  $v = v_0$ , where  $O(\Omega_0)$  denotes the  $Sp(4, \mathbb{R})$ -orbit containing the matrix  $\Omega_0$ , compare with [16].

Condition (b) means that the frequency map  $\omega : \mathbb{R}^p \rightarrow \mathbb{R}^m$  is submersive at  $v = v_0$  and the family  $\Omega(v)$  simultaneously is a versal unfolding of  $\Omega_0$  inside  $\mathfrak{sp}(4, \mathbb{R})$ . Since the co-dimension of the submanifold  $\{\omega(v_0)\} \times O(\Omega_0)$  in the space  $\mathbb{R}^m \times \mathfrak{sp}(4, \mathbb{R})$  is equal to  $m + 2$ , to ensure BHT non-degeneracy condition, we need at least  $m + 2$  parameters (i.e.  $p \geq m + 2$ ).

**Diophantine conditions.** The components of the vector  $\omega(v) \in \mathbb{R}^m$  are called the *internal frequencies* of the invariant torus  $T_v$ , and the positive imaginary parts of the eigenvalues of  $\Omega(v)$  are said to be the *normal frequencies*, see [20]. The vector  $\omega^N(v)$ , consisting of the normal frequencies, is referred to as the *normal frequency vector* at the torus  $T_v$ . Recall that the Floquet matrix  $\Omega(v_0)$  is in generic  $1 : -1$  resonance; in particular,  $\Omega(v_0)$  has a pair of double eigenvalues on the imaginary axis, say  $\pm i\lambda_0$  with  $\lambda_0 \neq 0$ , compare with figure 1. The number of the normal frequencies may depend on the parameter  $v$  in general, but we follow [16] and simply count  $\omega^N$  twice where appropriate.

**Lemma 4.1.** *There exists a reparametrization*

$$\phi : v \in \mathbb{R}^p \mapsto (\omega(v), \mu(v), \rho(v)) \in \mathbb{R}^m \times \mathbb{R}^2 \times \mathbb{R}^{p-m-2}$$

near  $v = v_0$  such that  $\mu(v_0) = (0, 0)$  and the normal frequency vector  $\omega^N(v)$  of the Hamiltonian  $X_{\omega, \mu, \rho} = X_{\phi(v)}$  is given by

$$\begin{cases} \omega^N = (\lambda_0 + \mu_1, \lambda_0 + \mu_1), & \text{for } \mu_2 \leq 0, \\ \omega^N = (\lambda_0 + \mu_1 + \sqrt{\mu_2}, \lambda_0 + \mu_1 - \sqrt{\mu_2}), & \text{for } \mu_2 > 0. \end{cases} \quad (24)$$

In particular, the family  $X$  always has two normal frequencies for all  $v$  in a sufficiently small neighbourhood of  $v_0 \in \mathbb{R}^p$ .

**Proof.** By the non-degeneracy condition and the inverse function theorem, there exists a reparametrization

$$\phi : v \in \mathbb{R}^p \mapsto (\omega(v), \mu(v), \rho(v)) \in \mathbb{R}^m \times \mathbb{R}^2 \times \mathbb{R}^{p-m-2}$$

such that the vector fields  $X_{\omega, \mu, \rho} = X_{\phi(v)}$  read

$$X_{\omega, \mu, \rho} = [\omega + O(|y|, |z|^2)] \frac{\partial}{\partial x} + [\Omega(\mu)z + O(|y|, |z|^2)] \frac{\partial}{\partial z},$$

where  $\Omega(\mu)$  is any universal unfolding of  $\Omega_0 = \Omega(v_0)$ . By [16, 51], the matrix family  $\Omega(\mu)$  can be chosen as

$$\Omega(\mu) = \begin{pmatrix} 0 & -\lambda_0 - \mu_1 & 1 & 0 \\ \lambda_0 + \mu_1 & 0 & 0 & 1 \\ -\mu_2 & 0 & 0 & -\lambda_0 - \mu_1 \\ 0 & -\mu_2 & \lambda_0 + \mu_1 & 0 \end{pmatrix}. \quad (25)$$

Now a direct calculation gives the normal frequencies (24) of the family  $X_{\omega, \mu, \rho}$ .  $\square$

The frequency map  $\mathcal{F} : \mathbb{R}^p \rightarrow \mathbb{R}^m \times \mathbb{R}^2$  is defined as  $\mathcal{F}(v) = (\omega(v), \omega^N(v))$ . We say that the frequency vector  $\mathcal{F}(v)$  satisfies the  $(\tau, \gamma)$ -Diophantine conditions if for a constant  $\tau > m - 1$  and for a positive parameter  $\gamma$ , we have

$$|\langle \omega(v), k \rangle + \langle \omega^N(v), \ell \rangle| \geq \gamma |k|^{-\tau}, \quad (26)$$

for all  $k \in \mathbb{Z}^m \setminus \{0\}$  and for all  $\ell \in \mathbb{Z}^2$  with  $|\ell_1| + |\ell_2| \leq 2$ . For a non-empty open (connected) neighbourhood  $U$  of  $v_0 \in \mathbb{R}^p$ , let  $\Gamma_{\tau, \gamma}(U)$  denote the set of parameters  $v \in U$  such that the internal and normal frequencies  $(\omega(v), \omega^N(v))$  satisfy the Diophantine condition (26). Also we need the subset  $\Gamma_{\tau, \gamma}(U')$ , where the set  $U' \subset U$  is given by

$$U' = \{v \in U : \text{dist}((\omega(v), \omega^N(v)), \partial \mathcal{F}(U)) > \gamma\}. \quad (27)$$

For  $\gamma$  sufficiently small, the set  $U'$  is still a non-empty open neighbourhood of  $v_0$ .

**Remark 4.2.** The set  $\Lambda$  of  $(\omega, \omega^N) \in \mathbb{R}^m \times \mathbb{R}^2$  subject to the Diophantine conditions (26) is a nowhere dense, uncountable union of closed half lines. The intersection  $\Lambda \cap \mathbb{S}^{m+1}$  with the unit sphere of  $\mathbb{R}^m \times \mathbb{R}^2$  is a closed set, which by Cantor–Bendixson theorem [43] is the union of a perfect set and a countable set. The complement of  $\Lambda \cap \mathbb{S}^{m+1}$  contains the dense set of resonant vectors  $(\omega, \omega^N)$ . Since all points in  $\Lambda \cap \mathbb{S}^{m+1}$  are separated by resonant hyperplanes, the perfect set is totally disconnected and hence a Cantor set. In  $\mathbb{S}^{m+1}$  this Cantor set tends to full Lebesgue measure as  $\gamma \downarrow 0$ . We refer to the set of frequencies  $(\omega(v), \omega^N(v))$  satisfying (26) as a ‘Cantor set’—a foliation of manifolds over a Cantor set.

**Theorem 4.3 (Quasi-periodic Hamiltonian Hopf bifurcation: persistence of Diophantine  $m$ -tori).** *Let  $H = H_v$  be the  $p$ -parameter real-analytic family of integrable Hamiltonians given by (19). Assume that*

- the torus  $T_{v_0}$  is in generic normal  $1 : -1$  resonance;
- the family  $H$  is non-degenerate at the invariant torus  $T_{v_0}$ .

*Then, for  $\gamma$  sufficiently small and for any  $p$ -parameter real-analytic family  $\tilde{H}$  of Hamiltonians on  $(M, \sigma)$  sufficiently close to  $H$  in the compact-open topology on complex analytic extensions, there exists a neighbourhood  $U$  of  $v_0 \in \mathbb{R}^p$  and a map*

$$\Phi : M \times U \longrightarrow M \times \mathbb{R}^p,$$

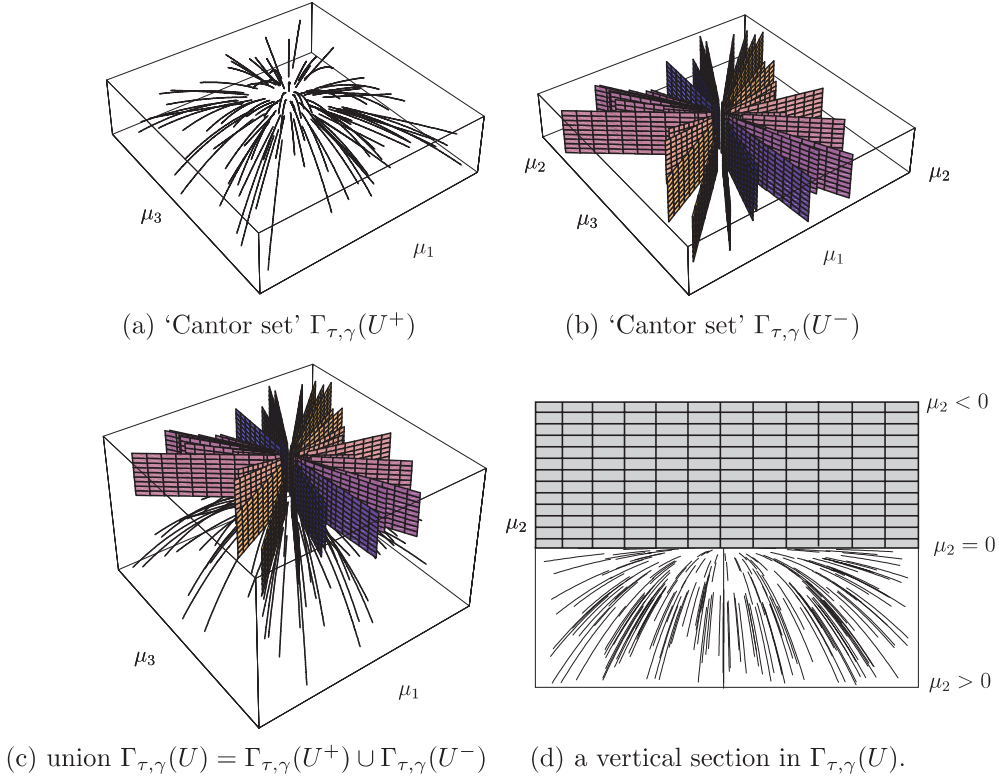
*defined near the normally resonant tori  $T_{v_0}$ , such that*

- (i)  $\Phi$  is a  $C^\infty$ -near-identity diffeomorphism onto its image;
- (ii) the image  $\Phi(V)$ , with  $V = \mathbb{T}^m \times \{y = 0\} \times \{z = 0\} \times \Gamma_{\tau, \gamma}(U')$ , is a Cantor family of  $X_{\tilde{H}}$ -invariant Diophantine tori, and the restriction of  $\Phi$  to  $V$  induces a conjugacy between  $H$  and  $\tilde{H}$ ;
- (iii) the restriction  $\Phi|_V$  is symplectic and preserves the (symplectic) normal linear part<sup>6</sup>  $NX = \omega(v)(\partial/\partial x) + \Omega(v)z(\partial/\partial z)$  of the family  $X = X_H$  of Hamiltonian vector fields associated with  $H$ .

In [16, 21] the results of theorem 4.3 are referred to as normal linear stability. For the proof see [16]. We refer to the Diophantine tori  $V$  (also its diffeomorphic image  $\Phi(V)$ ) as a *Cantor family of invariant  $m$ -tori*, as it is parametrized over a ‘Cantor set’.

Let us briefly discuss the geometrical pictures of the ‘Cantor set’  $\Gamma_{\tau, \gamma}(U)$ . In view of lemma 4.1, we may assume that  $v = (\omega, \mu, \rho)$  and that the normal frequencies are given

<sup>6</sup> For a discussion on symplectic normal linearization, see [20, 44].



**Figure 8.** Sketch of the 'Cantor sets'  $\Gamma_{\tau,\gamma}(U^+)$  and  $\Gamma_{\tau,\gamma}(U^-)$  corresponding to  $\mu_2 \geq 0$  and  $\mu_2 < 0$ , respectively. The total 'Cantor set'  $\Gamma_{\tau,\gamma}(U)$  is depicted in (c). The half planes in (a)–(c) give continua of invariant  $m$ -tori. In (d), a section of the 'Cantor set'  $\Gamma_{\tau,\gamma}(U)$ , along the  $\mu_2$ -axis, is singled out: the above grey region corresponds to a half plane given in (c)—a two-dimensional continuum of  $m$ -tori.

by (24). Moreover, we can suppress the parameter  $\rho$ , since the internal and normal frequencies do not depend on  $\rho$ . Denoting the subsets

$$U^+ = \{v \in U : \mu_2 > 0\},$$

$$U^- = \{v \in U : \mu_2 \leq 0\},$$

we have  $\Gamma_{\tau,\gamma}(U) = \Gamma_{\tau,\gamma}(U^+) \cup \Gamma_{\tau,\gamma}(U^-)$ . In the set  $\Gamma_{\tau,\gamma}(U^-)$ , the parameter  $\mu_2$  does not appear in the normal frequencies, so there are no Cantor gaps in the  $\mu_2$ -direction. It is not hard to see that the set  $\Gamma_{\tau,\gamma}(U^-)$  is the intersection of  $U^-$  and a union of closed half planes of the form  $\{(t\omega, t(\lambda_0 + \mu_1)) : t \geq 1\}$ , see figure 8(b). Similarly, from the Diophantine conditions (26) with  $\omega^N = (\lambda_0 + \mu_1 + \sqrt{\mu_2}, \lambda_0 + \mu_1 - \sqrt{\mu_2})$ , we see that the set  $\Gamma_{\tau,\gamma}(U^+)$  is a union of closed half parabolae of the form  $\{(t\omega, t(\lambda_0 + \mu_1), t^2\mu_2) : t \geq 1\}$ , see figure 8(a).

**Remark 4.4.** To ensure that the BHT non-degeneracy condition holds, we need at least  $m + 2$  parameters. Since we only have  $m$  distinguished or internal parameters, we are forced to use external parameters to satisfy the non-degeneracy condition, see the stability theorem 4.5. However, if we only care about the existence of invariant tori in the perturbed system, but refrain from explicitly relating perturbed invariant tori to unperturbed ones, we can use a much weaker form of non-degeneracy, the so-called Bakhtin–Rüssmann conditions, for details see [5, 19, 67].

For such a weaker persistence result, no external parameters will be required, see section 5.1. Also compare with [21].

**4.1.2. Application to the forced Lagrange top.** Now we turn back to the Lagrange top and consider the perturbation model (17) of the Lagrange top (near gyroscopic stabilization) coupled with a quasi-periodic oscillator. The unperturbed Hamiltonian  $\mathcal{H}_0$ , i.e. the Hamiltonian (17) with  $\varepsilon = 0$ , possesses invariant  $(n+1)$ -,  $(n+2)$ - and  $(n+3)$ -tori, see section 3. In this subsection, we investigate the persistence of the invariant  $(n+1)$ -tori of  $\mathcal{H}_0$  for  $\varepsilon \neq 0$ , applying theorem 4.3 with  $m = n+1$ . From (17) and (16), the Hamiltonian  $\mathcal{H}_\varepsilon$  has, in the local coordinates  $((q_1, \xi), (p_1, \eta), z) \in \mathbb{T}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^4$ , the form

$$\begin{aligned} \mathcal{H}_\varepsilon(q_1, \xi, p_1, \eta, z) = & \left( \frac{1}{2I_3}(a + p_1)^2 + c \right) + \frac{1}{8}((a + p_1)^2 - 4c)(z_1^2 + z_2^2) \\ & - \frac{1}{2}(a + p_1)(z_1 z_4 - z_2 z_3) + \frac{1}{2}(z_3^2 + z_4^2) + O(|z|^4) + G(\eta) + \varepsilon F(q_1, \xi, p_1, \eta, z, \varepsilon), \end{aligned} \quad (28)$$

where  $a/I_3$  is the frequency of the periodic solution  $\{(q_1, p_1, z) : p_1 = 0, z = 0\}$  in the unperturbed Lagrange top. The Hamiltonian vector field  $X^\varepsilon$  corresponding to (28), with respect to the 2-form  $dq_1 \wedge dp_1 + d\xi \wedge d\eta + dz^2$ , reads

$$\begin{cases} \dot{q}_1 = (a + p_1)/I_3 + O(|z|^2) + O(\varepsilon), \\ \dot{p}_1 = O(\varepsilon), \\ \dot{\xi}_1 = \delta_1(\eta) + O(\varepsilon), \\ \dot{\eta}_1 = O(\varepsilon), \\ \vdots \\ \dot{\xi}_n = \delta_n(\eta) + O(\varepsilon), \\ \dot{\eta}_n = O(\varepsilon), \\ \dot{z} = \Omega_{a,c} z + O(|p_1|, |z|^3) + O(\varepsilon), \end{cases} \quad (29)$$

where  $\Omega_{a,c} \in \mathfrak{sp}(4, \mathbb{R})$  is the Floquet matrix given by (14), and the vector  $(\delta_1(\theta), \dots, \delta_n(\theta))$  gives the frequencies of the quasi-periodic oscillator with  $\eta = \theta$ . Near the invariant torus given by  $(p_1, \eta, z) = (0, \theta, 0)$  of the unperturbed system  $X^0$ , we introduce the local coordinates  $(x, y) \in \mathbb{T}^{n+1} \times \mathbb{R}^{n+1}$  such that

$$x = (q_1, \xi) \in \mathbb{T} \times \mathbb{T}^n \quad \text{and} \quad y = (p_1, \eta - \theta) \in \mathbb{R} \times \mathbb{R}^n.$$

Also, we write

$$\omega = \omega(a, \theta) = (a/I_3, \delta_1(\theta), \dots, \delta_n(\theta)) \in \mathbb{R}^{n+1} \quad (30)$$

as the frequency vector of the invariant  $(n+1)$ -torus

$$\{(x, y, z) \in \mathbb{T}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^4 : (y, z) = (0, 0)\}$$

of the unperturbed system  $X^0$ . In the new coordinates  $(x, y, z)$ , the vector field  $X^\varepsilon$  obtains the form

$$\begin{aligned} X^\varepsilon(x, y, z) = & [\omega(a, \theta) + O(|y|, |z|^2) + O(\varepsilon)] \frac{\partial}{\partial x} + O(\varepsilon) \frac{\partial}{\partial y} \\ & + [\Omega_{a,c} z + O(|y|, |z|^3) + O(\varepsilon)] \frac{\partial}{\partial z}. \end{aligned} \quad (31)$$

For the family  $X^0$  to be BHT non-degenerate, see section 4.1, we need at least  $n+3$  parameters. Besides the distinguished parameters, i.e. conserved quantities  $(\theta, a) \in \mathbb{R}^n \times \mathbb{R}$ , we can incorporate the physical quantity  $c$  and the principal moment of inertia  $I_3$  as two extra parameters. In this way, the unperturbed system  $X^0$  becomes an  $(n+3)$ -parameter family of vector fields on  $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^4$  parametrized by

$$v = (\theta, a, c, I_3) \in \mathbb{R}^{n+3}.$$

Under such circumstances, we have the following stability theorem on the  $X^0$ -invariant isotropic  $(n+1)$ -tori given by  $(y, z) = (0, 0)$ .

**Theorem 4.5 (Forced Lagrange top: persistence of Diophantine  $(n+1)$ -tori).** *Let  $v_0 = (\theta_0, \pm 2\sqrt{c_0}, c_0, I_3^0) \in \mathbb{R}^{n+3}$ , where  $\theta_0, c_0, I_3^0$  are fixed and where  $c_0, I_3^0 > 0$ . Assume that the map*

$$\delta : \eta \in \mathbb{R}^n \mapsto (\delta_1(\eta), \dots, \delta_n(\eta)) \in \mathbb{R}^n \quad (32)$$

*is a local diffeomorphism at  $\eta = \theta_0$ . Then, for  $\gamma$  and  $|\varepsilon|$  sufficiently small, there exists a neighbourhood  $U$  of  $v_0$  in the parameter space  $\mathbb{R}^{n+3}$  such that the following holds: the nearly integrable perturbation  $X^\varepsilon$  has a Cantor family of invariant  $(n+1)$ -tori; these  $X^\varepsilon$ -invariant tori are  $C^\infty$ -near-identity diffeomorphic image of the tori  $\mathbb{T}^{n+1} \times \{y=0\} \times \{z=0\} \times \Gamma_{\tau, \gamma}(U')$ ; on these tori the diffeomorphism conjugates  $X$  and  $X^\varepsilon$  and preserves the normal linear behaviour.*

**Proof.** In view of theorem 4.3, we only need to check the non-degeneracy of the unperturbed system  $X^0$  at  $v_0 = (\theta_0, a_0, c_0, I_3^0)$ , where  $a_0 = \pm 2\sqrt{c_0}$ . First, the normal matrix  $\Omega_{a_0, c_0}$  is invertible for  $a_0 \neq 0$  and is in non-semisimple  $1 : -1$  resonance. By lemma 2.1, the matrix unfolding  $\Omega = \Omega_{a, c}$  is a (uni)versal unfolding of  $\Omega_{a_0, c_0}$  in  $\mathfrak{sp}(4, \mathbb{R})$ . Next, define the map  $\phi : \mathbb{R}^{n+3} \rightarrow \mathbb{R}^{n+1} \times \mathfrak{sp}(4, \mathbb{R})$  by

$$\phi(\theta, a, c, I_3) = (\omega(a, \theta), \Omega_{a, c}) = (a/I_3, \delta(\theta), \Omega_{a, c}),$$

then the derivative  $D_{v_0} \phi : \mathbb{R}^{n+3} \rightarrow \mathbb{R}^{n+1} \times \mathfrak{sp}(4, \mathbb{R})$  of  $\phi$  at  $v_0$  is given by

$$D_{v_0} \phi(\theta, a, c, I_3) = ((I_3^0 a - a_0 I_3)/(I_3^0)^2, D_{\theta_0} \delta(\theta), \tilde{\Omega}(a, c)),$$

where

$$\tilde{\Omega}(a, c) = \begin{pmatrix} 0 & \frac{a_0}{2} & 0 & 0 \\ -\frac{a_0}{2} & 0 & 0 & 0 \\ c_0 - \frac{a_0 a}{2} & 0 & 0 & \frac{a_0}{2} \\ 0 & c_0 - \frac{a_0 a}{2} & -\frac{a_0}{2} & 0 \end{pmatrix}.$$

Since the image satisfies  $\tilde{\Omega}(\mathbb{R}^2) = \ker \text{ad } \Omega_{a_0, c_0}^T$ , see also the proof of lemma 2.1, the map  $\phi = \omega \times \Omega$  is transversal to the submanifold

$$\{\omega(a_0, \theta_0)\} \times \text{ad } \Omega_{a_0, c_0}(\mathfrak{sp}(4, \mathbb{R})) = \{\omega(a_0, \theta_0)\} \times O(\Omega_{a_0, c_0}). \quad \square$$

**Remark 4.6.** Since the ‘Cantor set’  $\Gamma_{\tau, \gamma}(U')$  has full measure for  $\gamma \downarrow 0$ , a majority of all invariant  $(n+1)$ -tori survives small perturbations for  $\gamma$  sufficiently small, see [20, 44, 62].

#### 4.2. Persistence of the elliptic invariant $(m+1)$ -tori

Let us go back to the real-analytic family  $H$  of Hamiltonians of the form (19) with  $X_H$ -invariant  $m$ -tori  $T = \bigcup_v T_v$ , where

$$T_v = \{(x, y, z, v) \in \mathbb{T}^m \times \mathbb{R}^m \times \mathbb{R}^4 \times \mathbb{R}^p : y = 0, z = 0\}.$$

It is assumed that this family undergoes a quasi-periodic Hamiltonian Hopf bifurcation at  $v = v_0$ . This implies in particular that the torus  $T_{v_0}$  is normally  $1 : -1$  resonant. As said before, the phase space  $M = \mathbb{T}^m \times \mathbb{R}^m \times \mathbb{R}^4$ , near this torus, is foliated by  $X_H$ -invariant  $m$ -,  $(m+1)$  and  $(m+2)$ -tori. This local singular foliation gives a local stratification in a suitable parameter space described by a piece of the swallowtail catastrophe set as sketched in figure 4(a), compare with section 4.1. Our present aim is to study the persistence of the  $(m+1)$ -tori ‘surrounding’ the invariant  $m$ -tori of the unperturbed integrable system  $H$  given by (19). Our strategy follows [11, 14]: we first normalize the perturbation of the system  $H = H(y, z, v)$  as in (19), to obtain a truncated integrable part up to a suitable order; we show that this integrable truncation possesses a family of elliptic invariant  $(m+1)$ -tori; considering the non-integrable remainder in the normal form as a perturbation of the truncated integrable part, we apply the ‘standard’ KAM theory [20, 44, 56, 63] to yield the persistence of invariant elliptic  $(m+1)$ -tori. Again, as an example we consider the Lagrange top weakly coupled with a quasi-periodic oscillator.

**4.2.1. Nonlinear quasi-periodic Hamiltonian Hopf bifurcation.** Let  $\tilde{H} = \tilde{H}_v$  be a  $p$ -parameter real-analytic family of non-integrable Hamiltonians on  $M = \mathbb{T}^m \times \mathbb{R}^m \times \mathbb{R}^4 = \{x, y, z\}$  with the symplectic 2-form  $\sigma = \sum_{i=1}^m dx_i \wedge dy_i + \sum_{j=1}^2 dz_j \wedge dz_{j+2}$ . We investigate the quasi-periodic dynamics of the Hamiltonians  $\tilde{H}$  sufficiently close to the integrable family  $H$ , where  $H = H_v$  is given by

$$H_v(y, z) = \langle \omega(v), y \rangle + \frac{1}{2} \langle Jz, \Omega(v)z \rangle + O(|y|^2, |z|^3), \quad (33)$$

where  $J$  is given by (20). First of all, by theorem 4.3, the family  $\tilde{H}$  possesses a Cantor family of invariant  $m$ -tori; moreover, by preservation of normal linear behaviour, we have the following low-order normal form

$$\tilde{H}_v(x, y, z) = \langle \omega(v), y \rangle + \frac{1}{2} \langle Jz, \Omega(v)z \rangle + F(x, y, z, v), \quad (34)$$

where the parameters  $v$  are restricted to the ‘Cantor set’  $\Gamma_{\tau, y}(U')$  and where  $F = O(|y|^2, |z|^3)$ . To investigate the existence of the elliptic  $(m+1)$ -tori in the nearly integrable family  $\tilde{H}$ , we need to consider the higher order terms of  $\tilde{H}$ ; in other words, we need to normalize  $\tilde{H}$  up to higher orders. After applying the reparametrization  $v \rightarrow (\omega(v), \mu(v), \rho(v))$  as given in lemma 4.1, without loss of generality we can assume that  $v = (\omega, \mu, \rho) \in \mathbb{R}^m \times \mathbb{R}^2 \times \mathbb{R}^{p-m-2}$ . Then, following the proof of lemma 4.1—after a suitable change in coordinates—the family  $\tilde{H} = \tilde{H}_{\omega, \mu, \rho}(x, y, z)$  takes the form

$$\tilde{H}_{\omega, \mu, \rho}(x, y, z) = \langle \omega, y \rangle + \frac{1}{2} \langle Jz, \Omega(\mu)z \rangle + F(x, y, z, \omega, \mu, \rho), \quad (35)$$

where  $\Omega(\mu) \in \mathfrak{sp}(4, \mathbb{R})$  is given by

$$\Omega(\mu) = \begin{pmatrix} 0 & -\lambda_0 - \mu_1 & 1 & 0 \\ \lambda_0 + \mu_1 & 0 & 0 & 1 \\ -\mu_2 & 0 & 0 & -\lambda_0 - \mu_1 \\ 0 & -\mu_2 & \lambda_0 + \mu_1 & 0 \end{pmatrix}, \quad (36)$$

with constant  $\lambda_0 \neq 0$ . Notice that the normal frequency vector  $\omega^N = \omega^N(\mu_1, \mu_2)$  of the Hamiltonian family  $\tilde{H}$  is given by (24). For simplicity we will drop the parameter  $\rho$  for

our discussion below, since it can be easily incorporated again. In this way, we have that  $\nu = (\omega, \mu) \in \mathbb{R}^m \times \mathbb{R}^2$ .

As is done in [10, 11] for the (dissipative) quasi-periodic Hopf bifurcation, we start with normalizing higher order terms of the perturbed systems  $\tilde{H}_\nu$ . The idea is to remove non-integrable terms from lower order terms by applying suitable coordinate transformations; these transformations will be taken as the Hamiltonian time-one-flows generated by the Hamiltonian functions of the form  $\text{ad } S(L) = \{L, S\}$ , where  $S$  is the semisimple Hamiltonian

$$S = \langle \omega, y \rangle + (\lambda_0 + \mu_1)(z_1 z_4 - z_2 z_3)$$

and  $L$  a certain Hamiltonian function, for details see [appendix B](#). Here  $\{\cdot, \cdot\}$  stands for the Poisson bracket with respect to the standard 2-form  $\sigma$ . Under the above setting, we have the following nonlinear normal form theorem.

**Theorem 4.7 (Quasi-periodic normal form).** *Let  $H = H_\nu$  be the real-analytic family of integrable Hamiltonians given by (33). Assume that the assumptions of theorem 4.3 are satisfied and that the parameters  $\nu = (\omega, \mu, \rho)$  are restricted to  $\Gamma_{\tau, \gamma}(U')$ , see section 4.1, where  $U$  is a small neighbourhood of the fixed parameter  $\nu_0$ . Then, for any real-analytic Hamiltonian family  $\tilde{H} = \tilde{H}_\nu$  sufficiently close to  $H$  in the compact-open topology on complex analytic extensions, there exists a family of symplectic maps*

$$\Phi : \mathbb{T}^m \times \mathbb{R}^m \times \mathbb{R}^4 \times U \longrightarrow \mathbb{T}^m \times \mathbb{R}^m \times \mathbb{R}^4 \times \mathbb{R}^p$$

with the following properties. For fixed  $\nu$  the projection of  $\Phi$  is real-analytic in  $(x, y, z)$  and  $\Phi$  is  $C^\infty$ -near-identity. The Hamiltonian  $\tilde{H} \circ \Phi = G + R$  is decomposed into an integrable part  $G$  and a remainder  $R$ . The former reads

$$G = \langle \omega, y \rangle + (\lambda_0 + \mu_1)S + N + \mu_2 M + \frac{b}{2}M^2 + c_1 S M + c_2 S^2,$$

where  $S, N, M$  are the quadratic polynomials given by

$$S = z_1 z_4 - z_2 z_3, \quad N = \frac{1}{2}(z_3^2 + z_4^2) \quad \text{and} \quad M = \frac{1}{2}(z_1^2 + z_2^2) \quad (37)$$

and where the coefficients  $b, c_1$  and  $c_2$  depend on the parameters  $(\omega, \mu_1, \mu_2)$ . The remainder  $R$  satisfies

$$\frac{\partial^{q+|p|+|k|} R}{\partial^q \mu_2 \partial^p y \partial^k z}(x, 0, 0, \omega, \mu_1, 0) = 0,$$

for all indices  $(q, p, k) \in \mathbb{N}_0^3$  with  $2q + 4|p| + |k| \leq 4$ .

A proof of theorem 4.7 is given in [appendix B](#). For a similar nonlinear normal form theory in the dissipative or non-conservative setting see [10, 11] and in the Hamiltonian setting see [14, 61].

**Remark 4.8.**

1. The special case of theorem 4.7 for  $m = 1$  is considered in [61].
2. Theorem 4.7 holds in particular for the unperturbed system  $H$ , that is, for  $\tilde{H} = H$ . In this integrable case, the remainder  $R$  will be independent of the angle coordinates  $x$ ; moreover, the parameters  $\nu = (\omega, \nu, \rho)$  are not required to be restricted to the ‘Cantor set’ determined by the resonances.
3. The integrable part  $G$  in theorem 4.7 is said to be a normal form truncation of  $\tilde{H} \circ \Phi$  of order 4.

As the next step, we show that the integrable part  $G$  has a family of elliptic invariant  $(m+1)$ -tori. The invariant  $(m+1)$ -tori of the vector field  $X = X_G$ , associated with the Hamiltonian function  $G$ , are determined by the cubic equation

$$F(S, M) := S^2 - 4bM^3 - 4\mu_2 M^2 - 4c_1 S M^2 = 0 \quad (38)$$

in  $M$  and satisfy  $M > 0$ , compare with [28, 42, 51, 52]. This allows us to introduce the variable  $Q = -\frac{1}{2} \ln M$  conjugate to  $P = z_1 z_3 + z_2 z_4$  and the angle  $\varrho$  conjugate to  $S$  by means of

$$\begin{aligned} z_1 &= \frac{\sqrt{2} \cos \varrho}{\exp Q}, & z_3 &= \frac{S}{\sqrt{2}} \exp Q \sin \varrho + \frac{P}{\sqrt{2}} \exp Q \cos \varrho, \\ z_2 &= \frac{\sqrt{2} \sin \varrho}{\exp Q}, & z_4 &= -\frac{S}{\sqrt{2}} \exp Q \cos \varrho + \frac{P}{\sqrt{2}} \exp Q \sin \varrho, \end{aligned}$$

compare with [60, 61]. In these variables the symplectic structure reads

$$dx \wedge dy + d\varrho \wedge dS + dP \wedge dQ,$$

and the Hamiltonian  $G$  becomes

$$G = \langle \omega, y \rangle + (\lambda_0 + \mu_1)S + \frac{1}{4}e^{2Q}(P^2 + S^2) + \frac{b}{2}e^{-4Q} + \mu_2 e^{-2Q} + c_1 S e^{-2Q} + c_2 S^2. \quad (39)$$

**Proposition 4.9.** *The truncated integrable Hamiltonian  $G$  has an  $(m+1)$ -parameter family of elliptic invariant  $(m+1)$ -tori.*

**Proof.** The associated vector field  $X_G$  of  $G$  obtains the form

$$\begin{aligned} \dot{x} &= \omega, \\ \dot{\varrho} &= \lambda_0 + \mu_1 + \frac{1}{2}S e^{2Q} + c_1 e^{-2Q} + 2c_2 S, \\ \dot{y} &= 0, \\ \dot{S} &= 0, \\ \dot{P} &= \frac{1}{2}e^{2Q}[P^2 + F(S, e^{-2Q})], \\ \dot{Q} &= -\frac{1}{2}P e^{2Q}. \end{aligned}$$

We see that the family  $\mathcal{T} = \mathcal{T}_{s_0}$  given by

$$\begin{aligned} \mathcal{T}_{s_0} &= \{((x, \varrho), (y, S), P, Q) \in \mathbb{T}^{m+1} \times \mathbb{R}^{m+1} \times \mathbb{R}^2 : y = 0, S = s_0, P = 0 \\ &\text{and } F(s_0, e^{-2Q}) = 0\} \end{aligned} \quad (40)$$

forms an  $(m+1)$ -parameter family of invariant  $(m+1)$ -tori of  $X_G$ . By the inverse function theorem, the equation  $F(S, e^{-2Q}) = 0$  locally has a unique solution  $Q = Q(S)$  for any given small  $S$ , provided that  $b > 0$ . The latter condition is always satisfied in the supercritical case.

Next, we consider the torus  $\mathcal{T}_{s_0}$  given by  $(y, S, P, Q) = (0, s_0, 0, q_0)$ , where  $F(s_0, e^{-2q_0}) = 0$ . The normal matrix of  $X_G$  at  $\mathcal{T}_{s_0}$  is given by

$$\begin{pmatrix} 0 & 2e^{2q_0}(s_0^2 + 2be^{-6q_0}) \\ -\frac{1}{2}e^{2q_0} & 0 \end{pmatrix}, \quad (41)$$

which has only purely imaginary eigenvalues. This shows that the torus  $\mathcal{T}_{s_0}$  is indeed of the elliptic type.  $\square$



**Remark 4.10.** In the above proof, we see again that the cubic equation (38) indeed determines elliptic invariant tori of  $G$ . The Floquet matrix of the vector field  $X_G$  at the torus  $\mathcal{T}_{s_0}$  :  $(y, S, P, Q) = (0, s_0, 0, q_0)$  is given by (41). Hence, the normal frequency  $\omega^N(s_0)$  of  $\mathcal{T}_{s_0}$  is given by

$$\omega^N(s_0) = \sqrt{s_0^2 e^{4q_0} + 2be^{-2q_0}}. \quad (42)$$

Next we argue that the remainder term  $R$  in the normal form, as a function of  $(x, \rho, y, S, P, Q)$ , can be considered as a small perturbation of the truncated part  $G$ . First we localize near an invariant torus  $(y, S, P, Q) = (0, s_0, 0, q_0)$  and introduce coordinates  $(\zeta_1, \zeta_2) = (P, Q - q_0)$  and  $s = S - s_0$ . This leads to a symplectic transformation by means of  $\varrho \mapsto \varrho + P \cdot (\partial/\partial S)q_0$  and  $x_1 \mapsto x_1 + P \cdot (\partial/\partial y)q_0$ . In the new variables, we expand the Hamiltonian function  $G$  near the invariant torus  $(y, s, \zeta_1, \zeta_2) = (0, 0, 0, 0)$  to the Taylor series,

$$G = \langle \omega, y \rangle + \omega_{m+1}s + \left(\frac{1}{4}e^{2q_0} + c_2\right)s^2 + \frac{1}{4}e^{2q_0}\zeta_1^2 + (s_0^2 e^{2q_0} + 2be^{-4q_0})\zeta_2^2 + (s_0 e^{2q_0} - 2c_1 e^{-2q_0})s\zeta_2, \quad (43)$$

where the constant part and higher order terms are dropped and where

$$\omega_{m+1} = \lambda_0 + \mu_1 + \frac{1}{2}s_0 e^{2q_0} + c_1 e^{-2q_0} + 2c_2 s_0. \quad (44)$$

Furthermore, we write  $s_0 = |\mu_2|^{\frac{3}{2}}\delta$  whence  $\mu_2$  controls the approach to the origin in the  $(s_0, \mu_2)$ -plane and can be used as a scaling parameter<sup>7</sup> below. The next lemma gives an implicit dependence of  $q_0$  on  $\mu_2$  when  $\mu_2 \rightarrow 0$ .

**Lemma 4.11.** For  $\alpha = \frac{1}{2}e^{2q_0}\mu_2$  one has  $\alpha = O(1)$  as  $\mu \rightarrow 0$ .

**Proof.** Recall that  $F(s_0, e^{-2q_0}) = 0$ , implying that

$$2\delta^2\alpha^3 = b + 2\alpha + 2c_1\delta\sqrt{|\mu_2|\alpha}.$$

For  $\mu_2 \rightarrow 0$ , we have  $2\delta^2\alpha^3 - 2\alpha - b = 0$ ; it follows that the limit of  $\alpha$  is independent of  $\mu_2$ .  $\square$

**Theorem 4.12 (Quasi-periodic Hamiltonian Hopf bifurcation: persistence of Diophantine elliptic  $(m+1)$ -tori).** Let  $H = H_v$  be the family of integrable Hamiltonians given by (33) such that the assumptions in theorem 4.7 are satisfied. Then for  $\gamma$  sufficiently small and any real-analytic Hamiltonian family  $\tilde{H} = \tilde{H}_v$  sufficiently close to  $H$  in the compact-open topology on complex analytic extensions, there exists a map

$$\Phi : \mathbb{T}^m \times \mathbb{R}^m \times \mathbb{R}^4 \times U \longrightarrow \mathbb{T}^{m+1} \times \mathbb{R}^{m+1} \times \mathbb{R}^2 \times \mathbb{R}^p$$

changing coordinates near the normally resonant torus  $T_{v_0}$ , such that

- (i)  $\Phi$  is a  $C^\infty$ -near-identity diffeomorphism onto its image;
- (ii) the Hamiltonian  $\tilde{H} \circ \Phi^{-1}$  contains a Cantor family of elliptic invariant  $(m+1)$ -tori  $\tilde{V}$ ;
- (iii) on these tori  $\tilde{V}$  the normal linear part of the Hamiltonian vector fields associated with the family  $\tilde{H} \circ \Phi^{-1}$  is symplectically conjugate to Hamiltonian vector fields given by the truncated Hamiltonian function (43), provided that the parameters  $v = (\omega, \mu, \rho)$  and  $(\omega, \omega_{m+1}, s_0)$  satisfy the Diophantine conditions

$$|\langle \omega, k \rangle + k_{m+1}\omega_{m+1} + \ell\omega^N| \geq \gamma(|k| + |k_{m+1}|)^{-\tau}, \quad (45)$$

for all  $(k, k_{m+1}) \in \mathbb{Z}^m \times \mathbb{Z} \setminus \{(0, 0)\}$  and  $\ell = 0, \pm 1, \pm 2$ .

<sup>7</sup> Here the real parameter  $\delta$  is of order one for  $\mu_2 \rightarrow 0$ .

**Proof.** Writing  $\beta = 4\delta^2\alpha + b\alpha^{-2}$ , the coefficients of the terms  $\zeta_1^2$  and  $\zeta_2^2$  in the Taylor series (43) have the forms

$$\frac{\alpha}{2\mu_2} \quad \text{and} \quad \frac{\mu_2^2\beta}{2}, \quad (46)$$

respectively. By lemma 4.11, both the parameters  $\alpha$  and  $\beta$  are of order one as  $\mu_2$  approaches zero. The normal frequency of the truncated part  $G$  is of order  $\sqrt{|\mu_2|}$ , whence the scaling

$$(x, y, \varrho, s, \zeta_1, \zeta_2; \omega_{m+1}, \mu_2, \omega) \mapsto (x, \mu_2^{-\frac{9}{2}}y, \varrho, \mu_2^{-\frac{9}{2}}s, \mu_2^{-\frac{3}{2}}\zeta_1, \mu_2^{-3}\zeta_2; \mu_2^{-\frac{1}{2}}\omega_{m+1}, \mu_2, \mu_2^{-\frac{1}{2}}\omega)$$

together with a division of the integrable part  $G$  by  $\mu_2^5$  yields

$$\mu_2^{-5} \cdot G(x, y, \varrho, s, \zeta_1, \zeta_2) = \langle \omega, y \rangle + \omega_{m+1}s + \frac{\alpha}{2}\zeta_1^2 + \frac{\beta}{2}\zeta_2^2 + \dots$$

As  $\mu_2 \rightarrow 0$  the perturbation by both the higher order terms of  $G$  and the non-integrable remainder  $R$  is sufficiently small to yield the persistence of elliptic tori  $(\zeta_1, \zeta_2) = (0, 0)$  that satisfy the Diophantine conditions (45) on the internal frequencies  $\tilde{\omega} = (\omega, \omega_{m+1}) \in \mathbb{R}^m \times \mathbb{R}$  and the normal frequency  $\omega^N = \sqrt{\alpha\beta}$ , see [19].  $\square$

**4.2.2. Application to the forced Lagrange top.** We go back to the Lagrange top coupled with a quasi-periodic oscillator. Recall that the Lagrange top becomes gyroscopically stabilized by the supercritical Hamiltonian Hopf bifurcation, see the discussion in section 2. In this section, we apply the nonlinear analysis of the quasi-periodic Hamiltonian Hopf bifurcation, discussed in section 4.2.1, to study the local (nonlinear) dynamics of the perturbation model  $\mathcal{H}_\varepsilon$  of the Lagrange top with oscillator, see (17). In particular, we concentrate on the persistence of the elliptic  $(n+2)$ -tori in the unperturbed system  $\mathcal{H}_0$ , associated with the smooth part of the critical value set of the energy-momentum map of the Lagrange top, see figure 7. Following the discussion in section 4.1.2, there are local canonical coordinates  $(x, y, z) \in \mathbb{T}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^4$  such that

$$\mathcal{H}_\varepsilon(x, y, z) = \langle \omega(a, I_3, \theta), y \rangle + \frac{1}{2} \langle Jz, \Omega_{a,c}z \rangle + O(|y|, |z|^2) + O(\varepsilon),$$

where  $\omega(a, I_3, \theta) = (a/I_3, \delta(\theta)) \in \mathbb{R} \times \mathbb{R}^n$  and the Floquet matrix  $\Omega_{a,c} \in \mathfrak{sp}(4, \mathbb{R})$  is given by (14). Recall that  $a$  and  $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$  denote the angular-momentum values of the Lagrange top and of the (external) quasi-periodic oscillator, respectively; the  $(I_3, c) \in \mathbb{R}^2$  are physical parameters in the Lagrange top, see also section 2.1.

For  $\varepsilon$  sufficiently small, theorem 4.5 provides the normal linear behaviour of the Hamiltonian system  $\mathcal{H}_\varepsilon$  at invariant Diophantine  $(n+1)$ -tori. However, the quasi-periodic dynamics of the nearly integrable Hamiltonian  $\mathcal{H}_\varepsilon$  near the  $(n+1)$ -torus in normal  $1 : -1$  resonance is not completely determined by the normal linear part obtained by theorem 4.5, compare with section 4.2.1. Indeed, to inspect the existence of invariant quasi-periodic Floquet  $(n+2)$ -tori in the perturbed system  $\mathcal{H}_\varepsilon$ , we need to consider the fourth order terms in the (normalized) function  $\mathcal{H}_\varepsilon$ .

Now a nonlinear description of  $\mathcal{H}_\varepsilon$  is obtained by theorem 4.7; by this theorem, the Hamiltonian  $\mathcal{H}_\varepsilon$  can be normalized into the form

$$\begin{aligned} \mathcal{H}_\varepsilon = & \langle \omega(v), y \rangle + N + \frac{a}{2}S + \left( \frac{1}{4}a^2 - c \right)M + \frac{b(v, c, \varepsilon)}{2}M^2 + c_1(v, c, \varepsilon)SM \\ & + c_2(v, c, \varepsilon)S^2 + R_\varepsilon(x, y, z, v, c, \varepsilon), \end{aligned} \quad (47)$$

where  $v = (a, I_3, \theta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$  and where the remainder  $R_\varepsilon$  contains non-integrable terms—provided that the parameters  $(\omega, \mu_1, \mu_2) = (\omega, \frac{1}{2}a, \frac{1}{4}a^2 - c)$  satisfy the Diophantine

conditions (26). The polynomials  $M, S, N$  are given by (37). The (unperturbed) Lagrange top undergoes the supercritical Hamiltonian Hopf bifurcation, i.e. the coefficient function  $b(v, c, \varepsilon) > 0$  for all physical values of  $(v, c)$  and  $\varepsilon$  close to 0.

The normal form (47) holds in particular for the unperturbed integrable Hamiltonian  $\mathcal{H}_0$ , see remark 4.8(2). In this integrable case, no Diophantine conditions are required for the parameters  $(\omega, \mu_1, \mu_2)$ ; the remainder  $R_0$  consists of integrable terms being quasi-homogenous polynomials of degree  $s > 4$  in  $(y, S, M, \mu_2)$  with the weight  $(\alpha_y, \alpha_S, \alpha_M, \alpha_{\mu_2}) = (2, 1, 1, 1)$ .

Let  $X^0$  be the corresponding Hamiltonian vector field to  $\mathcal{H}_0$ . Following section 4.2.1, the vector field  $X^0$  has a family of elliptic invariant  $(n+2)$ -tori, which in the canonical coordinates  $(x, \varrho, y, S, (P, Q)) \in \mathbb{T}^{n+1} \times \mathbb{T}^1 \times \mathbb{R}^{n+1} \times \mathbb{R} \times \mathbb{R}^2$ , as introduced in section 4.2.1, read

$$\mathcal{T}_{s_0} = \{(x, \varrho, y, S, (P, Q)) : (y, S, P, Q) = (0, s_0, 0, q_0), F(s_0, e^{-2q_0}) = 0\},$$

where  $F$  is the cubic polynomial (38), compare with (40). In the local variables  $\tilde{x} = (x, \varrho) \in \mathbb{T}^{n+2}$ ,  $\tilde{y} = (y, S - s_0) \in \mathbb{R}^{n+2}$  and  $\tilde{z} = (P, Q - q_0) \in \mathbb{R}^2$  near the invariant tori  $\mathcal{T}_{s_0}$ , the normal linear part  $NX^0$  of  $X^0$  at the invariant  $(n+2)$ -tori  $\mathcal{T}_{s_0}$  obtains the form

$$NX^0 = \tilde{\omega}(v, a, s_0) \frac{\partial}{\partial \tilde{x}} + \tilde{\Omega}(s_0) \tilde{z} \frac{\partial}{\partial \tilde{z}}, \quad (48)$$

where  $\tilde{\omega} = (\omega(v), \omega_{n+2}(a, s_0))$  with  $\omega_{n+2} = -\frac{1}{2}a + \frac{1}{2}s_0 e^{2q_0} + c_1 e^{-2q_0} + 2c_2 s_0$  and where the Floquet matrix  $\tilde{\Omega}$  is given by

$$\tilde{\Omega}(s_0) = \begin{pmatrix} 0 & 2e^{2q_0}(s_0^2 + 2be^{-6q_0}) \\ -\frac{1}{2}e^{2q_0} & 0 \end{pmatrix} \in \mathfrak{sp}(2, \mathbb{R}),$$

see also the proof of proposition 4.9. Notice that, in the new local variables  $(\tilde{x}, \tilde{y}, \tilde{z})$ , the  $(n+2)$ -tori  $\mathcal{T}_{s_0}$  are given by  $(\tilde{y}, \tilde{z}) = (0, 0)$ . Considering  $(a, I_3, \theta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$  as parameters, we write  $\mathcal{T}_{a, I_3, \theta, s_0}$  for the tori  $\mathcal{T}_{s_0} : (\tilde{y}, \tilde{z}) = (0, 0)$ . Writing  $v = (a, I_3, \theta, s_0)$ , we use the parameters  $(a, I_3, \theta)$  to control the internal frequencies  $\tilde{\omega}(v) = (\omega(v), \omega_{n+2}(a, s_0))$ , while let  $s_0$  keep track of the normal frequency  $\omega^N = \sqrt{s_0^2 e^{4q_0} + 2be^{-2q_0}}$  at the torus  $\mathcal{T}_{a, I_3, \theta, s_0}$ . Recall that the frequencies of the quasi-periodic oscillator are given by  $\delta(\theta) = (\delta_1(\theta), \dots, \delta_n(\theta))$ , see section 3. Now suppose that the frequencies  $\delta$  depend diffeomorphically on the momentum value  $\theta$ , then the persistence of the  $(n+2)$ -torus family  $\bigcup \mathcal{T}_{a, I_3, \theta, s_0}$  is ensured by theorem 4.12 for the frequency vector  $(\omega, \omega_{n+2}, \omega^N)$  satisfying the Diophantine conditions (45). More precisely, we have the following theorem.

**Theorem 4.13 (Forced Lagrange top: persistence of Diophantine elliptic  $(n+2)$ -tori).**

Let  $v_0 = (a_0, I_3^0, \theta_0, s_0^0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$  be fixed, where  $s_0^0 \neq 0$  is small and  $I_3^0 > 0$ ,  $a_0 > 0$ . Assume that the map  $\delta : \theta \in \mathbb{R}^n \rightarrow \delta(\theta) \in \mathbb{R}^n$  is a local diffeomorphism at  $\theta = \theta_0$ . Then, for any sufficiently small  $\gamma$  and  $|\varepsilon|$ , there exists a neighbourhood  $U$  of  $v_0$  in the parameter space  $\mathbb{R}^{n+3}$  such that the following holds: the nearly integrable perturbation  $X^\varepsilon$  has a Cantor family of invariant  $(n+2)$ -tori; these  $X^\varepsilon$ -invariant tori are  $C^\infty$ -near-identity diffeomorphic image of the  $X^0$ -invariant tori

$$\bigcup_{v \in \Gamma_{\tau, \gamma}(U')} (\mathcal{T}_v \times \{v\});$$

on these tori the diffeomorphism conjugates  $X^0$  and  $X^\varepsilon$  and preserves the normal linear behaviour.

**Remark 4.14.** Instead of using  $a$  to control the internal frequency  $\omega_{n+2}$  and  $s_0$  to control the normal frequency  $\omega^N$ , we could have interchanged their roles. As in the case of the  $(n+1)$ -tori, see section 4.1.2, we need at least  $n+3$  parameters to ensure non-degeneracy

(i.e. submersivity) of the frequency map. Since we only have  $n + 2$  distinguished parameters which are  $(a, \theta, s_0) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ , we need the external parameter  $I_3$  or  $c$  as the extra parameter to obtain the persistence by theorem 4.12. Nevertheless, following [19, 20], we can get rid of the external parameter  $I_3$  or  $c$  by a rescaling in time, i.e. we do not need external parameters at all, thereby weakening our stability result: conjugacy between quasi-periodic dynamics on the invariant tori is replaced by equivalence. Here equivalence means diffeomorphisms mapping orbits into orbits with preservation of the time direction.

#### 4.3. Persistence of the invariant Lagrangian $(m + 2)$ -tori

In the previous two sections, we considered the persistence of the invariant  $m$ - and  $(m + 1)$ -tori of the integrable Hamiltonian  $H$  given by (19) from the singular torus foliation near the normally  $1 : -1$  resonant torus. Next to these tori we have invariant Lagrangian  $(m + 2)$ -tori in the local foliation. In this section, we investigate the persistence of these Lagrangian  $(m + 2)$ -tori. In the swallowtail stratification as sketched in figure 4, these tori are located at the open region (excluding the thread) above the surface. To yield the persistence, we apply the classical KAM theory [2, 46, 62] and the global version [13]. The Kolmogorov non-degeneracy condition—required for the KAM theory—on the Lagrangian tori near the resonant torus is provided by the presence of non-trivial monodromy, compare with [31, 57, 66]. As before we discuss the Lagrangian tori in the coupled system of the Lagrange top and an oscillator.

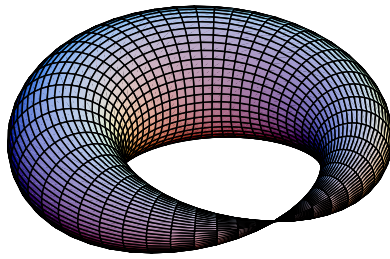
**4.3.1. Classical KAM theory.** We briefly recall the setting we work with. The phase space  $M$  is given by  $M = \mathbb{T}^m \times \mathbb{R}^m \times \mathbb{R}^4 = \{x, y, z\}$  with the 2-form  $\sigma = \sum dx_i \wedge dy_i + \sum dz_j \wedge dz_{j+1}$ . The space  $M$  admits a free  $\mathbb{T}^m$ -action, see section 1.1. We consider the integrable (i.e.  $\mathbb{T}^m$ -symmetric) Hamiltonian function  $H$  of the form

$$H(y, z, v) = \langle \omega(v), y \rangle + \frac{1}{2} \langle Jz, \Omega(v)z \rangle + F(y, z, v), \quad (49)$$

see also (19). The invariant torus  $T_{v_0}$  given by  $(y, z, v) = (0, 0, v_0)$  is in non-semisimple  $1 : -1$  resonance. For later use, we furthermore require that the derivative  $\partial F / \partial y$  has a maximal rank at the torus  $T_{v_0}$ .

As before we assume that  $H$  is invariant under the free  $\mathbb{S}^1$ -action generated by the polynomial  $S = \langle Jz, \Omega^s(v_0)z \rangle$ , where  $\Omega^s(v_0)$  denotes the semisimple part of  $\Omega(v_0)$ . Then, by the invariance with respect to this circle-action, the Hamiltonian  $H$  is Liouville-integrable with the  $m + 2$  first integrals  $\mathcal{EM} = (H, S, I)$ , where  $I : M \rightarrow \mathbb{R}^m$  is given by the integrals  $I = (y_1, \dots, y_m)$ . Recall from section 1.1, for each value  $a \in \mathbb{R}^m$ , that the Hamiltonian  $H$  induces a two-degrees-of-freedom Hamiltonian function  $H_a$  on the reduced phase space  $M_a = I^{-1}(a) / \mathbb{T}^m$ , identified with the space  $\mathbb{R}^4 = \{z_1, z_2, z_3, z_4\}$ . Associated with this reduced system, we have the reduced energy-momentum map  $EM_a : M_a \rightarrow \mathbb{R}^2$  defined by  $EM_a = (H_a, \hat{S})$ , where  $\hat{S} = z_1 z_4 - z_2 z_3$ . It is known from [24, 51] that the regular  $EM_a$ -fibre bundle—consisting of Lagrangian 2-tori—has non-trivial monodromy. This gives rise to non-triviality of the regular bundle. More precisely, let  $D$  be a punctured disc in the regular values set of  $EM_a$  which transversely intersects the thread (corresponding to the hyperbolic  $m$ -tori, see figure 4). Then, the foliation by  $EM_a$ -fibres over the boundary  $\partial D \cong \mathbb{S}^1$  of  $D$  forms a non-trivial bundle.

**Theorem 4.15 (Quasi-periodic Hamiltonian Hopf bifurcation: persistence of Diophantine Lagrangian  $(m + 2)$ -tori).** *Let  $H = H_v$  be the real-analytic family of Hamiltonians given by (49). Suppose that the Hessian of the higher order term  $F$  with respect to  $y$  is non-zero at the resonant torus  $T_{v_0}$ . Then there exists a neighbourhood  $U \subset \mathbb{R}^p$  of  $v_0$  such that for any real-analytic Hamiltonian  $\tilde{H}$  sufficiently close to  $H$  (with  $v \in U$ ) in the*



**Figure 9.** Pinched 2-torus with a circle ‘fibre’ pinched to an equilibrium  $p$ ; this invariant set consists of the stable and unstable manifolds of  $p$ , compare with [24].

*compact-open topology on complex analytic extensions the following holds: the perturbed Hamiltonian  $\tilde{H}$  has a Cantor family of invariant  $(m+2)$ -tori; this family is a  $C^\infty$ -near-identity diffeomorphic image of the family of maximal tori of  $H$  restricted to a ‘Cantor set’ (determined by the Diophantine conditions on the internal frequencies); on these tori, the diffeomorphism conjugates  $H$  and  $\tilde{H}$ .*

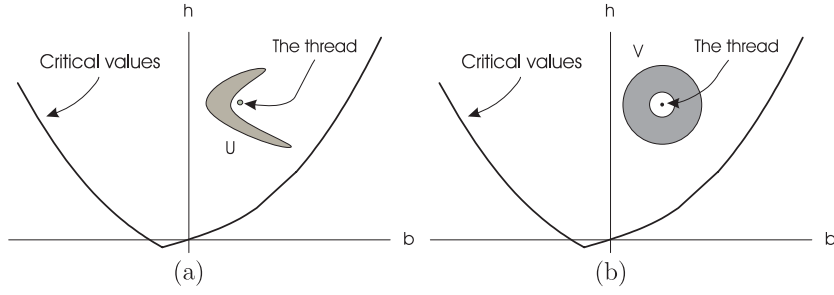
**Proof.** The Lagrangian  $(m+2)$ -tori of the integrable Hamiltonian  $H$  are the regular fibres of the energy-momentum map  $\mathcal{EM} : M \rightarrow \mathbb{R}^{m+2}$ . Our concern is with the persistence of these Lagrangian tori near the thread, see figure 4(a), when  $H$  is perturbed into a nearly integrable Hamiltonian family  $\tilde{H}$ . In view of the classical KAM theory [2, 46], most of these Lagrangian tori survive the perturbation in Whitney-smooth Cantor families. In the supercritical ((quasi)-periodic) Hamiltonian Hopf bifurcation, Kolmogorov non-degeneracy near the thread is a consequence of the non-trivial monodromy<sup>8</sup>. Indeed, non-degeneracy follows by an application of [31, 57, 66, 71] to the reduced two-degrees-of-freedom system  $H_a$  and the assumption that  $\det(\partial F / \partial y) \neq 0$  at the resonant torus  $T_{v_0}$ . We conclude that, for sufficiently small perturbation, the Lagrangian tori survive in a Whitney-smooth Cantor family of positive measure [23, 62].  $\square$

The corresponding KAM-conjugacies, which are only defined on locally trivial sub-bundles, can be glued together to provide a globally Whitney-smooth conjugacy from the integrable to the nearly integrable Cantor torus family [13]. Moreover, for sufficiently small perturbation this allows a proper extension of the non-trivial monodromy for the nearly integrable case. This may be of interest for semiclassical quantum mechanics, compare with [25, 26, 29, 32, 33, 69].

**Remark 4.16.** In two degrees of freedom the stable and unstable manifolds of a hyperbolic equilibrium with eigenvalue configuration as in figure 1 (also called a focus–focus point) often form a pinched 2-torus. A pinched 2-torus is homeomorphic to a one-point compactification of the cylinder  $\mathbb{S}^1 \times \mathbb{R}$ , see figure 9. The normally hyperbolic  $m$ -tori along the thread similarly lead to pinched  $(m+2)$ -tori.

**4.3.2. Application to the forced Lagrange top.** Now we apply the above ideas to the coupled system of the Lagrange top and a quasi-periodic oscillator, see the Hamiltonian model (17). Recall that the Lagrange top is a completely Liouville-integrable Hamiltonian system, giving rise to a lot of invariant 3-tori in the six-dimensional phase space. Indeed, by the Liouville–Arnold theorem [1, 4], each regular fibre of the energy-momentum map  $\mathcal{EM} = (H, \mathcal{M}^r, \mathcal{M}^l)$ ,

<sup>8</sup> For more details on monodromy see [24, 30, 58].



**Figure 10.** (a) Cross section of a smoothly contractible domain  $U$  cut by the plane  $a = \text{const.}$  (b) Non-contractible domain  $V$  around the thread.

see section 2, is a regular 3-torus. Recall that in the image space  $\mathbb{R}^3 = \{(h, a, b)\}$  of the map  $\mathcal{EM}$ , see figure 7, the regular values (associated with the regular 3-tori) of  $\mathcal{EM}$  form the open set (excluding the thread) above the two-dimensional critical surface  $\mathcal{C}$ .

Adding the quasi-periodic oscillator with  $n$  frequencies, the invariant regular 3-tori of the top become invariant regular  $(n + 3)$ -tori, which are Lagrangian tori of the integrable Hamiltonian system  $\mathcal{H}_0 = H_{\text{loc}} + G$ , see (17). As usual, the persistence of these tori will be our main issue. To this end, we need to check the Kolmogorov condition in the Hamiltonian system  $\mathcal{H}_0$ . Recall that close to gyroscopic stabilization, regular 3-tori of the top live next to the singular (or pinched) 3-tori of the top associated with critical values at the thread, compare with figure 7. These pinched 3-tori are related to the focus–focus singularities of the once reduced Hamiltonian  $H_a$ , see (9).

Let  $\mathcal{B}$  be a Lagrangian torus bundle over a smoothly contractible open domain  $U$  (close to the thread) in the regular values set of  $\mathcal{EM}$ , compare with figure 10(a). Then, this bundle contains no non-trivial monodromy, see [30]. By the Liouville–Arnold theorem [1, 4], in the whole neighbourhood  $\mathcal{B}$  of an invariant regular  $(n + 3)$ -torus, one can introduce angle-action coordinates  $(x, y) \in \mathbb{T}^{n+3} \times \mathbb{R}^{n+3}$  such that the symplectic form is given by  $dx \wedge dy$  and  $y_1, \dots, y_{n+3}$  are commuting first integrals of  $\mathcal{H}_0$ . Moreover, the function  $\mathcal{H}_0$  depends only on the action variable  $y$  and the frequencies of the system are given by  $\omega_1 = \partial \mathcal{H}_0 / \partial y_1(y), \dots, \omega_{n+3} = \partial \mathcal{H}_0 / \partial y_{n+3}(y)$ . Local dynamics of  $\mathcal{H}_0$  on the regular  $(n + 3)$ -tori in  $\mathcal{B}$  determined by the level sets

$$\{(y_1, \dots, y_{n+3}) = \text{const}\}$$

is given by the trajectories of the vector field

$$X(x, y) = \sum_{i=1}^{n+3} \omega_i \frac{\partial}{\partial x_i}.$$

Now by the classical KAM theory [2, 46, 62], a Cantor subfamily (of full measure) of these tori is persistent under a small non-integrable perturbation, if the Jacobian matrix

$$\left\{ \frac{d\omega_i}{dy_j} \right\}_{1 \leq i, j \leq n} = \left\{ \frac{\partial^2 \mathcal{H}_0}{\partial y_i \partial y_j} \right\}_{1 \leq i, j \leq n} \quad (50)$$

is non-singular (i.e. the Kolmogorov non-degeneracy condition holds). This condition implies that the frequencies  $\omega_1, \dots, \omega_{n+3}$  are fully (at least locally) controlled by the action variable  $y$ . In such a case, the latter serves as distinguished parameter of our system.

For the coordinates  $(x, y)$ , we may write

$$\begin{aligned} x &= (q_1, q_2, q_3, \xi_1, \dots, \xi_n) \in \mathbb{T}^{n+3}, \\ y &= (p_1, p_2, p_3, \eta_1, \dots, \eta_n) \in \mathbb{R}^{n+3}, \end{aligned}$$

where  $(q, p) \in \mathbb{T}^3 \times \mathbb{R}^3$  and  $(\xi, \eta) \in \mathbb{T}^n \times \mathbb{R}^n$  denote the action-angle variables in the Lagrange top and in the quasi-periodic oscillator, respectively. In particular, the variables  $q_1$  and  $p_1$ , as before, are the rotation angle and the angular momentum about the figure axis of the top, respectively, while  $(q_2, q_3, p_2, p_3)$  are action-angle coordinates of the once reduced system  $H_a$ , see section 2.

**Remark 4.17.** One of the action-variables  $p_2$  and  $p_3$  can be chosen to be the angular momentum map  $m_a : (u, v) \in \mathcal{R}_a \mapsto v_3 \in \mathbb{R}$  being induced by the angular momentum  $\mathcal{M}^l$  (to the once reduced phase space  $\mathcal{R}_a$ ) along the vertical axis, see section 2. In this case, we have  $p_1 = a = \mathcal{M}^r$ ,  $p_2 = b = \mathcal{M}^l$  and  $p_3 = p_3(H_{\text{loc}}, \mathcal{M}^r, \mathcal{M}^l)$ . Notice that  $\partial H_{\text{loc}} / \partial p_3 = \omega_3 \neq 0$ ; it follows that the values of  $(p_1, p_2, p_3)$  are (locally) completely determined by the distinguished parameters  $(h, a, b) = (H_{\text{loc}}, \mathcal{M}^r, \mathcal{M}^l)$ .

We assume that the frequency map  $\delta = \partial \mathcal{H}_0 / \partial \eta : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a diffeomorphism. In this case, it remains to check the Kolmogorov condition for the three-degrees-of-freedom Hamiltonian  $H_{\text{loc}}$ , see (16), of the Lagrange top near gyroscopic stabilization. Notice that the rotation frequency  $\omega_1 = a/I_3$ , see section 2; it follows that the Kolmogorov condition in the reduced Hamiltonian  $H_a$ , see (9), yields that in the full system  $H_{\text{loc}}$  of the unperturbed Lagrange top. At this point, we can use the results of [57, 66], adapted to the present case: let the angular momentum value  $a$  be fixed, if a singular torus foliation of the energy-momentum map  $(H_a, m_a) : \mathcal{R}_a \rightarrow \mathbb{R}^2$  contains a singular fibre of *focus-focus* or *complex-saddle* type, then the Kolmogorov condition holds for  $H_a$  in a full neighbourhood of this singular fibre. Next, we recall some basics of focus-focus singularities from [35, 58]: a critical point of  $\mathcal{EM}_a$  is of the focus-focus type, if a neighbourhood of this singularity admits symplectic coordinates  $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in \mathbb{R}^4$ , such that the symplectic form restricted to this neighbourhood is given by  $d\zeta_1 \wedge d\zeta_3 + d\zeta_2 \wedge d\zeta_4$  and that the quadratic parts  $[H_a]_2$  and  $[m_a]_2$  of  $H_a$  and  $m_a$ , respectively, obtain the form

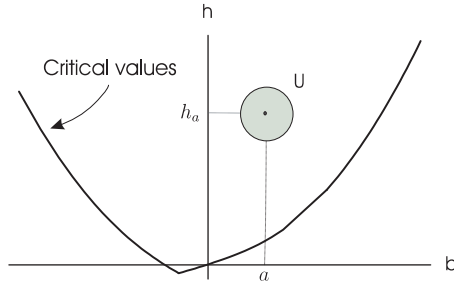
$$\begin{aligned} [H_a]_2 &= \alpha_1(\zeta_1\zeta_3 + \zeta_2\zeta_4) + \beta_1(\zeta_1\zeta_4 - \zeta_2\zeta_3), \\ [m_a]_2 &= \alpha_2(\zeta_1\zeta_3 + \zeta_2\zeta_4) + \beta_2(\zeta_1\zeta_4 - \zeta_2\zeta_3), \end{aligned} \tag{51}$$

for some real constants  $\alpha_1, \alpha_2, \beta_1$  and  $\beta_2$  with  $\alpha_1\beta_2 \neq \alpha_2\beta_1$ . Notice that the Floquet exponents of the Hamiltonian  $H_a$  are given by  $\pm\alpha_1 \pm i\beta_1$ , compare with figure 1. Now a singular fibre of  $\mathcal{EM}_a$  is said to be of the focus-focus type, if it contains a single singularity which is of the focus-focus type, see [57].

**Lemma 4.18.** *For a given angular momentum value  $a$  with  $0 < |a| < 2\sqrt{c}$ , let  $h_a = c + a^2/(2I_3)$ . Then, the fibre  $\mathcal{EM}_a^{-1}(h_a, a)$  is a singular focus-focus fibre. Hence, under such a circumstance, the Kolmogorov non-degeneracy condition holds on the invariant regular 2-tori close to the singular fibre  $\mathcal{EM}_a^{-1}(h_a, a)$ .*

**Proof.** Since  $\mathcal{EM}_a(p_+(a)) = (h_a, a)$ , where  $p_+(a) = (0, 0, 1, 0, 0, a) \in \mathcal{R}_a$  is a critical point of  $\mathcal{EM}_a$ , the fibre  $\mathcal{EM}_a^{-1}(h_a, a) \subseteq \mathbb{R}^4$  is singular: a pinched 2-torus, compare with figure 9. Moreover, the point  $p_+(a)$  is the only singularity in  $\mathcal{EM}_a^{-1}(h_a, a)$ . It remains to show that  $p_+(a)$  is a focus-focus singularity of  $\mathcal{EM}_a$ . To this end, we use the local forms of  $H_a$  and  $m_a$  at the singularity  $p_+(a)$ , introduced in section 2. Recall that in suitable Darboux coordinates  $z \in \mathbb{R}^4$  near  $p_+(a)$ , the quadratic part of  $H_a$  is given by the 2-jet in (13), while that of  $m_a$





**Figure 11.** Critical value set of the reduced energy-momentum map  $\mathcal{EM}_a$ , where  $0 < |a| < 2\sqrt{c}$ . The isolated critical value is located at  $(h, b) = (h_a, a) = (c + (1/2I_3)a^2, a)$ .

is given by  $z_1 z_4 - z_2 z_3$ . These two quadratic parts generate a Cartan subalgebra of the Lie algebra of quadratic forms under Poisson bracket, which can be verified by identifying the Lie algebra of quadratic forms with the matrix Lie algebra  $\mathfrak{sp}(4, \mathbb{R})$ . By [73] and the fact that the eigenvalues of the quadratic part in (13) are hyperbolic for  $0 < |a| < 2\sqrt{c}$ , there exists a linear symplectic change of coordinates such that the quadratic parts of  $H_a$  and  $m_a$  obtain the forms (51) with  $\alpha_1 = \frac{1}{2}\sqrt{4c - a^2}$ ,  $\beta_1 = \frac{1}{2}a$ ,  $\alpha_2 = 0$  and  $\beta_2 = 1$ .  $\square$

The critical value set of the map  $\mathcal{EM}_a$ , for a given value  $a$ , is the intersection of the plane  $a = \text{const}$  and the critical value set of the energy-momentum map  $\mathcal{EM}$ , see figure 7. In the case where  $0 < |a| < 2\sqrt{c}$ , such a section is sketched in figure 11. Observe that the isolated critical value  $(h_a, a)$  corresponds to the point  $(h_a, a, a) \in \mathbb{R}^3$  at the thread. Now by lemma 4.18, the three-degrees-of-freedom Hamiltonian function  $H_{\text{loc}}$  satisfies the Kolmogorov non-degeneracy condition on regular 3-tori close to each point at the thread with  $|a| < 2\sqrt{c}$ . Since the model  $H_{\text{loc}}$  of the Lagrange top is real-analytic and the determinant of (50) is not constant, the latter condition can be continued to the case where  $|a| \geq 2\sqrt{c}$ .

To formulate the persistence, we need to impose Diophantine conditions on the frequencies  $\omega = (\omega_1, \dots, \omega_{n+3})$ : for fixed  $\tau > n + 2$  and a parameter  $\gamma > 0$ , we require that

$$|\langle \omega, k \rangle| \geq \gamma |k|^{-\tau} \quad (52)$$

for all  $k \in \mathbb{Z}^{n+3} \setminus \{0\}$ . As before we denote for an open set  $V$  of frequencies by  $\Gamma_{\tau, \gamma}(V') \subseteq V$  the subset of  $\omega \in V$  satisfying (52) and bounded away from the boundary  $\partial V$ . Once the Kolmogorov non-degeneracy holds, the internal frequency vector  $\omega = \omega(h, a, b, \theta)$  is completely determined by the distinguished parameters  $(h, a, b, \theta)$ . Now a direct application of the standard KAM theory [2, 19, 23, 62] yields the following.

**Theorem 4.19 (Forced Lagrange top: persistence of Diophantine Lagrangian  $(n+3)$ -tori).**

Let the parameter  $v_0 = ((h_0, a_0, b_0), \theta_0) \in \mathbb{R}^3 \times \mathbb{R}^n$  be fixed, where  $(h_0, a_0, b_0)$  is a regular value of the energy-momentum map  $\mathcal{EM}$ . Assume that the map

$$\eta \in \mathbb{R}^n \mapsto (\delta_1(\theta), \dots, \delta_n(\theta)) \in \mathbb{R}^n \quad (53)$$

is a local diffeomorphism at  $\eta = \theta_0$ . Then for sufficiently small  $|\varepsilon|$  there exists a neighbourhood  $U$  of  $v_0$  in  $\mathbb{R}^3 \times \mathbb{R}^n$  such that the following holds: the family  $X^\varepsilon$  has a Cantor family of invariant  $(n+3)$ -tori which is a  $C^\infty$ -near-identity diffeomorphic image of  $X^0$ -invariant  $(n+3)$ -tori

$$\bigcup_{\omega(v) \in \Gamma_{\tau, \gamma}(V')} (\mathbb{T}^{n+3} \times \{y = v\}),$$

where  $v = (a, b, h, \eta) \in U$  and  $V = \omega(U)$ ; restricted to these tori the diffeomorphism conjugates  $X^0$  and  $X^\varepsilon$  and preserves the normal linear behaviour.



**Remark 4.20.**

1. While theorems 4.5 and 4.13 only address invariant tori near the gyroscopic stabilization, the present results hold for all regular Lagrangian tori near the thread and thus exceed a mere application of theorem 4.15.
2. Theorem 4.19 deals with the persistence of Lagrangian tori as fibres of a trivial sub-bundle of  $(n + 3)$ -tori. This is associated with the situation in figure 10(a). The question is what happens to regular fibres of non-trivial sub-bundles, compare with the situation sketched in figure 10(b), after small perturbations. As said before, by [13] local KAM-conjugacies obtained by theorem 4.19 can be glued together to yield a global conjugacy. This allows an extension of non-trivial monodromy in the integrable Lagrangian torus bundle to the non-integrable one. In this sense, we may say that the non-trivial monodromy in the integrable torus bundle survives a nearly integrable perturbation.

**5. Conclusions**

We considered a family of  $\mathbb{T}^m$ -symmetric Hamiltonians on the phase space  $M = \mathbb{T}^m \times \mathbb{R}^m \times \mathbb{R}^4$  with an invariant normally  $1 : -1$  resonant torus. As the parameter varies, the torus changes from (normally) hyperbolic to (normally) elliptic. This generically gives rise to a quasi-periodic Hamiltonian Hopf bifurcation, see section 1.1. Near the normally resonant torus the phase space  $M$  is the union of elliptic  $m$ -tori, elliptic  $(m + 1)$ -tori, Lagrangian  $(m + 2)$ -tori and the pinched  $(m + 2)$ -tori formed by the (un)stable manifolds of hyperbolic  $m$ -tori. This singular torus foliation gives a stratification in a suitable parameter space: the strata are determined by the dimension of the tori. The local geometry of this stratification is a piece of the swallowtail catastrophe set: the  $m$ -tori are located at the one-dimensional part (the crease and the thread),  $(m + 1)$ -tori at the regular part of the surface and  $(m + 2)$ -tori at the open region above the surface, see figure 4(a). By the previous KAM results [2, 20, 44, 46, 56, 62, 63] and the normal linear stability [16], these tori survive in Whitney-smooth Cantor families, under small nearly-integrable perturbations, compare with figure 4(b). In view of the global KAM theory [13], the non-trivial monodromy in the integrable Lagrangian torus bundle can be extended to the nearly integrable case. Concerning the parameter domains regarding the Diophantine tori of the different dimensions  $m, m + 1$  and  $m + 2$ , these are attached to one another in a Whitney-smooth way, as suggested by the integrable approximation. Following [15] we speak of a Cantor stratification. Here the stratum of the  $(m + 1)$ -tori consists of density points of  $(m + 2)$ -quasi-periodicity of large  $(2m + 4)$ -dimensional Hausdorff measure. Similarly, the stratum of the  $m$ -tori consists of density points of  $(m + 1)$ -quasi-periodicity of large  $(2m + 2)$ -dimensional Hausdorff measure. Compare also with [19].

As a leading example we studied the Lagrange top near gyroscopic stabilization weakly coupled to a quasi-periodic oscillator. In particular, we considered the existence of quasi-periodic invariant tori in the perturbed Hamiltonian  $\mathcal{H}_\varepsilon$  given by (17). Our setting was rather general, since we allowed for all forms of perturbations (i.e. couplings). In such a general case where both rotational symmetries  $\Phi^r$  and  $\Phi^l$  of the top were broken, we found Cantor families of invariant Diophantine tori of dimensions  $n + 1$ ,  $n + 2$  and  $n + 3$  in the perturbed systems, which are slight deformations of the unperturbed tori.

Next, we make some remarks about the reduction in the number of parameters. In addition, we give a few comments on the models of the special perturbations given in section 3. Finally, we address two open problems.

### 5.1. Fewer parameters

For the persistence of invariant isotropic tori, that is, for theorems 4.5, 4.12 and 4.19, we required the BHT non-degeneracy conditions as formulated in section 4.1. On the one hand, such conditions yield rather strong persistence: not only the ‘majority’ of the invariant tori survives small perturbations, but also the corresponding normal linear data of the unperturbed system  $\mathcal{H}_0$  survives. In particular, the preservation of internal and normal frequencies holds. On the other hand, these generic conditions require a lot of parameters. This causes a lack of distinguished (or internal) parameters available in our system (e.g.  $a$ ,  $b$ ,  $h$  and  $\theta$ ) and forces us to use external parameters (e.g.  $I_3$  and  $c$ ), compare with remarks 4.4 and 4.14. To avoid using external parameters, we may ask for a weaker persistence result, in which we are only interested in the existence of a Whitney-smooth Cantor family of invariant tori in the perturbed system, compare with [19, 67]. Notice that for the persistence of the Lagrangian  $(n+3)$ -tori, we did not need any external parameters. Next, we consider the reduction in the (external) parameters in the persistence of the invariant  $(n+1)$ -tori and the invariant  $(n+2)$ -tori. Recall that in the latter case, the unperturbed family  $X^0$  is parametrized by the  $n+3$  parameters  $(\theta_0, s_0, a, I_3)$ , where  $(\theta_0, s_0, a) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$  are distinguished parameters, while the parameter  $I_3 \in \mathbb{R}$  is the only external one. Instead of wishing to control all (internal and normal) frequencies as required by the BHT non-degeneracy condition, here we just want to keep track of the frequency ratios. The least number of parameters controlling these ratios is  $n+2$ , implying that we can get rid of one of the parameters, which we choose to be the external parameter  $I_3$ . Following [19, 20, 44], a similar (but weaker) persistence result such as theorem 4.12 holds for the unperturbed Hamiltonian system  $X^0$  considered as an  $(n+2)$ -parameter family parametrized over  $(\theta_0, s_0, a)$ , compare with remark 4.14. For this weaker persistence, the stable  $(n+2)$ -tori projected into the frequency space form a Cantor set of large measure, compare with [19, 20, 44].

To achieve the stability theorem 4.5 of the invariant  $(n+1)$ -tori, we are forced to use the two external parameters  $(I_3, c)$  to ensure the BHT non-degeneracy condition. As said before in remark 4.4, we can get rid of these two parameters by using the weaker non-degeneracy condition of Bakhtin–Rüssmann [5, 19, 21, 67, 68], thereby weakening the stability theorem 4.5. Let us consider the unperturbed system  $X^0$  as the family parametrized over  $(a, \theta_0) \in \mathbb{R} \times \mathbb{R}^n$ , that is, only  $(a, \theta_0)$  are considered as parameters. Recall from section 4.2 that the internal frequencies of the invariant tori  $\eta = \theta$  are given by  $\omega(a, \theta) = (a/I_3, \delta(\theta))$ , see (30), and the two parameters in the linear centralizer unfolding are  $\lambda_0 + \mu_1 = \frac{1}{2}a$  and  $\mu_2 = c - \frac{1}{4}a^2$ .

The family  $X^0 = X_{a, \theta_0}^0$  would satisfy the Bakhtin–Rüssmann condition at  $(a, \theta) = (a_0, \theta_0)$ , if the following were satisfied: there exists a neighbourhood  $U \subset \mathbb{R}^{n+1}$  of  $(a_0, \theta_0)$  in the parameter space and a positive integer  $N$  such that the image of the map  $(\omega, \mu) : U \rightarrow \mathbb{R}^{n+3}$  does not lie in any  $(n+2)$ -dimensional subspace. Since both  $\omega_1$  and  $\mu_1$  depend only linearly on  $a$ , this cannot be achieved. In this sense the Lagrange top is a degenerate integrable system.

### 5.2. Extra symmetries

In this subsection, we discuss Cantor families of surviving invariant Diophantine tori under perturbations where at least one of the two symmetries  $\Phi^r$  and  $\Phi^l$  still remains, compare with the models listed in section 3. In the general case, the Cantor families of stable KAM-tori contain continua of tori (the projection of these continua into the parameter space has no Cantor gaps). We refer to these projections as *continuous structures* in the Cantor family of surviving tori. In the special cases of symmetry-preserving perturbations, it seems natural to expect extra continuous structures. We distinguish between three special cases as mentioned

in section 3: the Lagrange top with a horizontal oscillator, with a ‘circular’ oscillator about the figure axis and with a vertical oscillator.

- (i) With a horizontal oscillator: the base point of the top is attached to two linear oscillators which are placed in a horizontal plane. In this case, the perturbation term  $F$  in the perturbed system (17) is independent of the angle variable  $q_1 \in \mathbb{R}/(2\pi\mathbb{Z})$  denoting the rotational angle about the figure axis. Notice that the left symmetry  $\Phi^l$  is not preserved. Dividing out the circle symmetry  $\Phi^r$  about the figure axis reduces the number of the degree of freedom in the perturbed system  $\mathcal{H}_\varepsilon$  by one. Consequently, we are left with two internal frequencies to be controlled. Hence, the (external) parameter  $I_3$  which is used for keeping track of the internal frequency  $\omega_1 = a/I_3$ , see (30), is no longer of importance in the stability theorems 4.5 and 4.12. This means in particular that we can get rid of the parameter  $I_3$  without weakening our stability results, which is not always possible under a general perturbation, compare with section 5.1. If we still keep  $I_3$  as a parameter, then the Cantor family of surviving Diophantine tori continues along the  $I_3$ -direction, that is, there are no Cantor gaps in the  $I_3$ -direction. This corresponds to the fact that any rotation of any frequency about the figure axis is independent of a horizontal forcing. Finally, following the discussion of section 5.1, we can in theorem 4.5 also eliminate the parameter  $c$  by a time-scaling, resulting in a weaker stability result that, however, no longer needs external parameters.
- (ii) With a ‘circular’ oscillator: in this case, the top is coupled via a circular spring to a quasi-periodic oscillator. Here the right symmetry (about the figure axis) is affected, while the left one (of the vertical axis) is preserved under perturbation. So far we used the local model  $H_{\text{loc}}$  given by (16), which is based on the reduction of the right symmetry everywhere. For the present discussion, however, it is convenient to have a local Hamiltonian model based on the reduction of the left symmetry. As is done in [24], using this left reduction, we end up with a slightly different local model. We will not discuss this in detail and only point out the main differences. First of all, the internal frequency  $\omega_1$  becomes  $b/I_3$  instead of  $a/I_3$ , where  $b$  denotes the value of the angular momentum along the vertical axis. Secondly, the Floquet matrix  $\Omega$  reads

$$\Omega = \begin{pmatrix} 0 & b(I_3^{-1} - \frac{1}{2}) & 1 & 0 \\ -b(I_3^{-1} - \frac{1}{2}) & 0 & 0 & 1 \\ c - \frac{b^2}{4} & 0 & 0 & b(I_3^{-1} - \frac{1}{2}) \\ 0 & c - \frac{b^2}{4} & -b(I_3^{-1} - \frac{1}{2}) & 0 \end{pmatrix}, \quad (54)$$

compare with Floquet matrix (14). Now we can repeat the arguments given in (i), thereby concluding that the parameter  $I_3$  can be ignored in our consideration of the persistence results. In particular, for a circular forcing where the left symmetry is preserved, the Cantor families of Diophantine tori have continuous structure in the  $I_3$ -direction.

- (iii) A vertical oscillator: the top is coupled to a quasi-periodic oscillator which acts vertically. In this case, both the left and the right symmetries are preserved under perturbation. Hence, we can reduce the perturbed system  $\mathcal{H}_\varepsilon$  to a system with  $n + 1$  degrees of freedom by carrying out two divisions of the symmetries  $\Phi^r$  and  $\Phi^l$ , where  $n$  denotes the number of frequencies of the oscillator. This suggests that the number of parameters which is at least needed for our persistence results is  $n + 1$ . Indeed, following (i) and (ii), we can get

rid of  $I_3$  from the right symmetry and  $c$  from the left symmetry. Consequently, all Cantor families of surviving invariant tori are continuous with respect to the external parameters  $I_3$  and  $c$ .

### 5.3. Quasi-periodic response solutions

Here we consider the Lagrange top with a quasi-periodic forcing of  $n \geq 2$  frequencies. As before we are interested in the situation where the top is close to gyroscopic stabilization, compare with section 3. We assume that the perturbed system remains Hamiltonian and is given by the Hamiltonian function

$$H_\varepsilon = H_0 + \varepsilon f(\delta_1 t, \dots, \delta_n t, \dots),$$

where  $H_0$  denotes the Hamiltonian of the unperturbed Lagrange top and  $f$  is a quasi-periodic function in  $t$  with the frequencies  $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{R}^n$ . The function  $f$  may depend on the perturbation parameter  $\varepsilon$  and other variables as well. The quasi-periodicity of  $f$  means that the frequencies  $\delta_1, \dots, \delta_n$  of the forcing are rationally independent.

Let us briefly reconsider the unperturbed Lagrange top near gyroscopic stabilization. Following section 2.3, there are suitable symplectic coordinates  $(q_1, p_1, z) \in \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}^4$  such that the unperturbed Hamiltonian  $H_0$  takes the local form

$$H_0(q_1, p_1, z) = \left( \frac{1}{2I_3}(a + p_1)^2 + c \right) + \frac{1}{8}((a + p_1)^2 - 4c)(z_1^2 + z_2^2) - \frac{1}{2}(a + p_1)(z_1 z_4 - z_2 z_3) + \frac{1}{2}(z_3^2 + z_4^2) + O(|z|^4),$$

compare with (16). Recall that the parameter  $a$  is the angular momentum value associated with the rotation about the figure axis; the constant  $I_3$  is a principal moment of inertia of the Lagrange top and  $c > 0$  is a physical constant depending on the position of the centre of mass. We remark that near gyroscopic stabilization  $|a|$  is close to the critical value  $a_0 = 2\sqrt{c}$ , see section 2.3.1. In the local coordinates mentioned above, the perturbed Hamiltonian  $H_\varepsilon$  now takes the form

$$H_\varepsilon = H_0(q_1, p_1, z) + \varepsilon f(\zeta, q_1, p_1, z; \varepsilon),$$

where the variables  $\zeta = (\delta_1 t, \dots, \delta_n t) \in \mathbb{T}^n$ . Next we consider the function  $H_\varepsilon$  as a Hamiltonian on the phase space  $M = \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}^4 = \{\zeta, \eta, q_1, p_1, z\}$ , where  $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$  denote the action-variables conjugated to the angle variables  $\zeta$ . Now the  $(n+3)$ -degrees-of-freedom Hamiltonian vector field  $X^\varepsilon = X_{H_\varepsilon}$  associated with  $H_\varepsilon$ , with respect to the symplectic 2-form  $\sigma = d\zeta \wedge d\eta + dq_1 \wedge dp_1 + dz_1 \wedge dz_3 + dz_2 \wedge dz_4$ , reads

$$X^\varepsilon : \begin{cases} \dot{\zeta} = \delta, \\ \dot{\eta} = O(\varepsilon), \\ \dot{q}_1 = a/I_3 + O(|p_1|, |z|^2) + O(\varepsilon), \\ \dot{p}_1 = O(\varepsilon), \\ \dot{z} = \Omega_{a,c} z + O(|p_1|, |z|^2) + O(\varepsilon), \end{cases} \quad (55)$$

compare with the vector field (21). Recalling from section 4, the integrable unperturbed system  $X^0$  undergoes a quasi-periodic Hamiltonian Hopf bifurcation as the angular momentum value  $a$  passes through the critical value  $a_0$ . At the bifurcation we have an  $X^0$ -invariant normal  $1 : -1$  resonant  $(n+1)$ -torus given by  $\eta = \text{const}$ ,  $p_1 = 0$  and  $z = 0$ . Near this torus, the phase space locally is singularly foliated by  $X^0$ -invariant hyperbolic and elliptic  $(n+1)$ -tori, elliptic  $(n+2)$ -tori and Lagrangian  $(n+3)$ -tori, compare with section 4. As before, we ask for

the persistence of the invariant tori of all dimensions in this singular foliation for small  $\varepsilon$ ; but now we keep the frequencies  $\delta_1, \dots, \delta_n$  of the forcing fixed. We refer to this as a quasi-periodic response problem of the Lagrange top, compare with [10, 14, 19, 56, 70]. As usual, we need to require Diophantine conditions on the internal and normal frequencies of the unperturbed Hamiltonian system  $X^0$ , see section 4.1.

Let us first consider the multi-frequency  $\delta$  of the forcing as a parameter. Then the present problem is a special case of the perturbation model (21) studied in section 4. Hence, the invariant  $(n+1)$ -,  $(n+2)$ - and  $(n+3)$ -tori of the singular foliation near the resonant torus survive in Whitney-smooth Cantor families. These surviving tori give a Cantor stratification<sup>9</sup> in a suitable parameter space, see figure 4(b).

It turns out that the mentioned conclusions remain true for the response problem mentioned above, where the frequencies  $\delta_1, \dots, \delta_n$  of the forcing are constant. Indeed, the proof of the normal linear stability [16] in the symplectic setting shows that for the perturbation  $X^\varepsilon$  the parameter-shifting in the  $\delta$ -direction can be chosen as the identity map, compare also with [10, 14]. As mentioned before, the unperturbed Lagrange top has two rotational symmetries associated with the rotations about the figure axis and the vertical axis. Since the preservation of these symmetries depends only on the way of acting by the forcing on the Lagrange top and not on the frequencies of the forcing, the discussion in section 5.2 remains valid in this response problem. Finally, we observe that the ‘Cantor set’ of parameters associated with the stable Diophantine tori differs from that in the varying forcing-frequencies case: fixing the frequencies amounts to intersecting the full ‘Cantor set’ with the affine subspace  $\eta = \text{const.}$

#### 5.4. Open problems

In this final subsection, we address two problems related to the quasi-periodic Hamiltonian Hopf bifurcation and the Lagrange top.

**5.4.1. Subcritical case.** So far we have considered the supercritical case of quasi-periodic Hamiltonian Hopf bifurcations, regarding the persistence of the local singular foliation by invariant isotropic tori near the bifurcation point, see section 4. Our analysis showed that such a foliation survives small perturbations. For similar persistence results in the subcritical case see [41]. This is important for a better understanding of the hydrogen atom in crossed electric and magnetic fields [32, 33], where the subcritical bifurcation occurs. The singular torus foliation for this case is described by the tail of the swallowtail surface<sup>10</sup>. The approach for the supercritical case also works in the subcritical case, where furthermore a subordinate quasi-periodic centre-saddle bifurcation occurs.

**5.4.2. Downwards spinning top.** As an example of the supercritical Hamiltonian Hopf bifurcation, we concentrated on a local perturbation problem of a vertically upwards spinning top close to gyroscopic stabilization. The same problem can be considered for a vertically downwards spinning top and a similar approach can be developed. Since this motion is always (Lyapunov) stable, the local dynamics of such a top is less involved. Indeed, let us go back to the once reduced Hamiltonian  $H_a$  and to the phase space  $\mathcal{R}_a$ , see section 2. A vertically downwards spinning top is associated with the critical point  $p_- = (0, 0, -1, 0, 0, -a) \in \mathcal{R}_a$ .

<sup>9</sup> The strata are determined by the dimension of invariant tori.

<sup>10</sup> Complementary to the part of figure 4(a).

There exist local canonical coordinates  $z = (z_1, z_2, z_3, z_4)$  near  $p_-$  in which the Hamiltonian function  $H_a$  obtains the form:

$$H_a(z) = \frac{1}{8I_1}[(9a^2 + 4cI_1)(z_1^2 + z_2^2) + 12(z_1z_4 - z_2z_3) + 4(z_3^2 + z_4^2)] + \dots,$$

where we dropped the constant term and where the dots denote the higher order terms. As before, we assume that  $I_1 = 1$ ,  $c > 0$  and  $a \neq 0$ . In the local coordinates  $z \in \mathbb{R}^4$ , the linear part of the Hamiltonian  $H_a$  is given by

$$\Omega = \frac{1}{4} \begin{pmatrix} 0 & -\mu_1 & 4 & 0 \\ \mu_1 & 0 & 0 & 4 \\ -\mu_2 & 0 & 0 & -\mu_1 \\ 0 & -\mu_2 & \mu_1 & 0 \end{pmatrix},$$

where  $\mu_1 = 6a$  and  $\mu_2 = 9a^2 + 4c$ . The eigenvalues of  $\Omega$  are of the forms  $\pm i(\mu_1 \pm 2\sqrt{\mu_2})$ , which are purely imaginary and simple. Hence, for the perturbation problem where the Lagrange top is parametrically forced (with  $n$  frequencies in the forcing), see section 3, ‘classical’ KAM theory is expected to yield the persistence of the elliptic invariant  $(n+1)$ -tori. It is known from, e.g. [4, 22, 59] that the lower equilibrium of a 2D pendulum with a parametric forcing is destabilized by a parametric resonance. Similarly, this resonance may destroy the stable downwards spinning motions of a top. We ask the following question: does parametric resonance occur in the gaps in the ‘Cantor set’ given by KAM theory? Knowing [22, 59], we expect a positive answer. However, in contrast to [22, 59] where destabilization of an equilibrium is considered, here we deal with periodic solutions.

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### Appendix A. Proof of theorem 4.7

In this section, we give a proof of theorem 4.7. Let us recall the setting from section 4.2.1 briefly. Our phase space is the symplectic manifold  $M = \mathbb{T}^m \times \mathbb{R}^m \times \mathbb{R}^4 = \{x, y, z\}$  with the canonical 2-form  $\sigma = \sum_{i=1}^m dx_i \wedge dy_i + \sum_{j=1}^2 dz_j \wedge dz_{j+2}$ . We consider a  $p$ -parameter family  $\tilde{H} = \tilde{H}(x, y, z, v)$  of nearly integrable Hamiltonian functions of the form

$$\tilde{H}(x, y, z, v) = \langle \omega(v), y \rangle + \frac{1}{2} \langle Jz, \Omega(v)z \rangle + F(x, y, z, v), \quad (56)$$

where  $\Omega(v) \in \mathfrak{sp}(4, \mathbb{R})$ ,  $J$  is as in (20) and  $F$  denotes the higher order terms. Moreover, the Floquet matrix  $\Omega_0 = \Omega(v_0)$  is in non-semisimple  $1 : -1$  resonance and the family  $\tilde{X} = X_{\tilde{H}}$  of Hamiltonian vector fields associated with  $\tilde{H}$  satisfies the BHT non-degeneracy condition, see section 4.1.1. Then, following the discussion in section 4.2.1, we can assume that  $v = (\omega, \mu) \in \mathbb{R}^m \times \mathbb{R}^2$  and that the Floquet matrices  $\Omega(v)$  are given by (25). Under such circumstances, the family  $\tilde{H}$  takes the following shape:

$$\tilde{H}_v(x, y, z) = \langle \omega, y \rangle + N + \mu_1 S + \mu_2 M + F(x, y, z, \omega, \mu), \quad (57)$$

where  $N = \frac{1}{2}(z_3^2 + z_4^2)$ ,  $S = (z_1z_4 - z_2z_3)$ ,  $M = \frac{1}{2}(z_1^2 + z_2^2)$ . Recall that the normal frequency vector  $\omega^N(v)$  of the Hamiltonian system  $\tilde{H}_v$  is given by (24) and that the parameters

$v = (\omega, \mu_1, \mu_2)$  are subject to the  $(\tau, \gamma)$ -Diophantine condition (26). We assume that the multi-parameter  $(\omega, \mu_1, \mu_2)$  is in a neighbourhood  $U \subseteq \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}$  of  $v_0 = (\omega_0, \mu_1^0, 0)$ .

Our aim now is to construct a coordinate transformation bringing the family  $\tilde{H} = \tilde{H}_v(x, y, z)$  as in (57) into a simpler form—a Birkhoff-like normal form [7, 39]. Our approach follows [10, 11, 61]. For a similar normalization of vector fields near a periodic orbit see [22].

#### Appendix A.1. Preliminaries

We introduce the complex variables  $(\zeta, \eta) \in \mathbb{C}^2$  as follows:

$$\zeta = \frac{1}{\sqrt{2}}(z_1 + iz_2) \quad \text{and} \quad \eta = \frac{1}{\sqrt{2}}(z_3 - iz_4),$$

compare with [11, 51, 61]. Notice that the symplectic form  $\sigma$  now becomes  $\sigma = dx \wedge dy + d\zeta \wedge d\bar{\zeta} + d\eta \wedge d\bar{\eta}$ . We first fix the parameters  $(\omega, \mu_1)$  and write  $\hat{z} = (\zeta, \bar{\zeta}, \eta, \bar{\eta})$ . Developing the Hamiltonian  $\tilde{H}$  into the Taylor–Fourier series in the new variables  $(x, y, \hat{z}, \mu_2)$  near the torus  $\{(y, \hat{z}) = (0, 0)\}$ , we obtain

$$\tilde{H} = \sum_{k \in \mathbb{Z}^n} \sum_{\ell, p, q} H_{k, \ell, p, q} \mu_2^\ell y^p \zeta^{q_1} \bar{\zeta}^{q_2} \eta^{q_3} \bar{\eta}^{q_4} e^{i\langle x, k \rangle}, \quad (58)$$

where  $\ell \in \mathbb{Z}_{\geq 0}$ ,  $p = (p_1, \dots, p_m) \in \mathbb{Z}_{\geq 0}^m$ ,  $q = (q_1, q_2, q_3, q_4) \in \mathbb{Z}_{\geq 0}^4$  and where  $y^p = y_1^{p_1} \dots y_m^{p_m}$ . In the following, we refer to the integer  $r = 2\ell + 4|p| + |q|$  as the *degree* of a function of the form  $f(x, y, \hat{z}, \mu_2) = f_0(x) \mu_2^\ell y^p \hat{z}^q$ . We denote by  $\mathcal{F}$  the space of the Taylor–Fourier series in  $(x, y, \hat{z}, \mu_2)$  of the form (58) and  $\mathcal{F}_r \subset \mathcal{F}$  the subspace consisting of the Taylor–Fourier series as in (58) but quasi-homogeneous with degree  $r$ . For later use, we also introduce, for the given non-negative integers  $(\alpha, \beta)$ , the subspaces

$$\mathcal{F}_{\alpha, \beta}^{k, \ell, p} = \left\{ F \in \mathcal{F}_r : F = \sum_{(s, t) = (0, 0)}^{(\alpha, \beta)} F_{s, t} \mu_2^\ell y^p \zeta^s \bar{\zeta}^t \eta^{\beta-t} \bar{\eta}^{\alpha-s} e^{i\langle x, k \rangle} \right\},$$

where the indices  $(k, \ell, p)$  are fixed, compare with [61]. Observe that  $\mathcal{F}_{\alpha, \beta}^{k, \ell, p}$  is a subspace of  $\mathcal{F}_{2\ell+4|p|+\alpha+\beta}$  and that the following decompositions hold:

$$\mathcal{F} = \bigoplus_{j \geq 2} \mathcal{F}_j, \quad \text{with } \mathcal{F}_j = \bigoplus_{\substack{2\ell+4|p|+\alpha+\beta=j \\ k \in \mathbb{Z}^n}} \mathcal{F}_{\alpha, \beta}^{k, \ell, p} \quad (59)$$

It is known from, e.g. [1, 4, 12] that the set of all Hamiltonian functions on  $(M, \sigma)$  equipped with the standard Poisson bracket  $\{\cdot, \cdot\}$  (with respect to the 2-form  $\sigma$ ) forms a Lie algebra. For a given function  $F$ , the adjoint operator  $\text{ad } F : \mathcal{F} \rightarrow \mathcal{F}$  is defined as  $\text{ad } F(G) = \{G, F\}$ .

Next we will focus on the map  $\text{ad } \tilde{H}_2$  associated with the function

$$\tilde{H}_2(x, y, \hat{z}; \omega, \mu_1) = \langle \omega, y \rangle + \eta \bar{\eta} + i\mu_1(\zeta \eta - \bar{\zeta} \bar{\eta}), \quad (60)$$

since we will use this function to perform the normalization procedure on  $\tilde{H}_\mu$ . Notice that

$$\text{ad } \tilde{H}_2 = \omega \frac{\partial}{\partial x} + \left( \bar{\eta} \frac{\partial}{\partial \zeta} + \eta \frac{\partial}{\partial \bar{\zeta}} \right) + i\mu_1 \left( \zeta \frac{\partial}{\partial \zeta} - \bar{\zeta} \frac{\partial}{\partial \bar{\zeta}} - \eta \frac{\partial}{\partial \eta} + \bar{\eta} \frac{\partial}{\partial \bar{\eta}} \right). \quad (61)$$

In the following, for any given  $F \in \mathcal{F}_2$ , the restriction of  $\text{ad } F$  to the subspace  $\mathcal{F}_r$  is denoted by  $\text{ad}_r F$  for  $r \geq 2$ .

**Lemma A.1.** *The subspaces  $\mathcal{F}_r$  and  $\mathcal{F}_{\alpha, \beta}^{k, \ell, p}$  are invariant under both  $\text{ad } \tilde{H}_2$  and  $\text{ad } \hat{S}$ , where  $\hat{S} = \langle \omega, y \rangle + i\mu_1(\zeta \eta - \bar{\zeta} \bar{\eta})$ . Moreover, the eigenvalues of the operator  $\text{ad } \hat{S}$  are of the form*

$$i\langle \omega, k \rangle + i\mu_1(\alpha - \beta)$$

*for all  $k \in \mathbb{Z}^m$  and all non-negative integers  $\alpha$  and  $\beta$ .*



**Proof.** Let  $F$  be a polynomial in  $\mathcal{F}_{\alpha,\beta}^{k,\ell,p}$ . Without loss of generality, we can assume that  $F = \mu_2^\ell y^p \zeta^s \bar{\zeta}^t \eta^{\beta-t} \bar{\eta}^{\alpha-s} e^{i(k,x)}$ . A brief calculation gives

$$\text{ad } \tilde{H}_2(F) = \left[ i\langle k, \omega \rangle + i\mu_1(\alpha - \beta) + \left( s \frac{\bar{\eta}}{\zeta} + t \frac{\eta}{\bar{\zeta}} \right) \right] F$$

and

$$\text{ad } \hat{S}(F) = [i\langle k, \omega \rangle + i\mu_1(\alpha - \beta)]F, \quad (62)$$

which belong to  $\mathcal{F}_{\alpha,\beta}^{k,\ell,p}$ . In view of the decompositions (59), the spaces  $\mathcal{F}_r$  are  $\text{ad } \tilde{H}_2$ - and  $\text{ad } \hat{S}$ -invariant. The eigenvalues of  $\text{ad } \hat{S}$  are read off from (62).  $\square$

**Remark A.2.** The function  $\hat{S}$  is the semisimple part of  $\tilde{H}_2$ , while the nilpotent part is  $\hat{N} = \eta\bar{\eta}$ .

For any given order  $r \geq 2$ , by lemma A.1 and the Diophantine conditions (26), the subspace  $\ker \text{ad}_r \hat{S} \subseteq \mathcal{F}_r$  is given by

$$\ker \text{ad}_r \hat{S} = \bigoplus_{2\ell+4|p|+2\alpha=r} \mathcal{F}_{\alpha,\beta}^{0,\ell,p}. \quad (63)$$

Under these circumstances, we have the direct sum decomposition

$$\mathcal{F}_r = \ker \text{ad}_r \hat{S} \oplus \text{im } \text{ad}_r \hat{S}. \quad (64)$$

Denote by  $\mathcal{R}$  the subspace of  $\mathcal{F}$  containing all real valued power series, i.e.

$$\mathcal{R} = \{F \in \mathcal{F} : \bar{F} = F\}.$$

Similarly, we introduce real linear subspaces  $\mathcal{R}_r, \mathcal{R}_{\alpha,\beta}^{\ell,p,k}$  corresponding to  $\mathcal{F}_r, \mathcal{F}_{\alpha,\beta}^{\ell,p,k}$ , respectively. It is readily checked that  $\mathcal{R}, \mathcal{R}_r$  and  $\mathcal{R}_{\alpha,\beta}^{\ell,p,k}$  all are invariant under  $\text{ad } \hat{S}$ , implying that decompositions (63) and (64) also hold when  $\text{ad}_r \hat{S}$  is restricted to  $\mathcal{R}_r$ .

**Lemma A.3.** Suppose that the internal frequencies  $\omega$  and the normal frequency  $\mu_1$  satisfy the Diophantine conditions (26). Then, for any given integer  $r \geq 2$  and for any  $R \in \text{im } \text{ad}_r \hat{S}$ , there exists a unique formal solution  $F \in \text{im } \text{ad}_r \hat{S}$  such that  $\text{ad}_r \tilde{H}_2(F) = R$ . In particular, we have  $\text{im } \text{ad}_r \hat{S} \subseteq \text{im } \text{ad}_r \tilde{H}_2$ .

**Proof.** Each set  $\mathcal{F}_r$  is decomposed into  $\text{ad } \tilde{H}_2$ -invariant subspaces  $\mathcal{F}_{\alpha,\beta}^{k,\ell,p}$ , where  $2\ell+4|p|+\alpha+\beta=r$ ; we may restrict everything to such a subspace  $\mathcal{F}_{\alpha,\beta}^{k,\ell,p}$  for fixed indices  $(k, \ell, p, \alpha, \beta)$ . Hence, without loss of generality, we assume that  $R$  has the quasi-homogeneous form

$$R = \sum_{(s,t)=(0,0)}^{(\alpha,\beta)} R_{s,t} \mu_2^\ell y^p \zeta^s \bar{\zeta}^t \eta^{\beta-t} \bar{\eta}^{\alpha-s} e^{i(k,x)} \quad (65)$$

and look for  $F$  in that same form. Then,

$$\text{ad } \tilde{H}_2(F) = \sum_{(s,t)=(0,0)}^{(\alpha,\beta)} \left[ \lambda_k + \left( s \frac{\bar{\eta}}{\zeta} + t \frac{\eta}{\bar{\zeta}} \right) \right] F_{s,t} \mu_2^\ell y^p \zeta^s \bar{\zeta}^t \eta^{\beta-t} \bar{\eta}^{\alpha-s} e^{i(k,x)},$$

where  $\lambda_k = i\langle \omega, k \rangle + i\mu_1(\alpha - \beta)$ . By splitting (64) and the fact that  $\ker \text{ad}_r \hat{S}$  is  $\text{ad } \tilde{H}_2$ -invariant, the solution  $F$  of the equation  $\text{ad } \tilde{H}_2(F) = R$  can be chosen in  $\text{im } \text{ad}_r \hat{S}$ . Now the equation



ad  $\tilde{H}_2(F) = R$  gives

$$R_{s,t} = \lambda_k F_{s,t} + (s+1)F_{s+1,t} + (t+1)F_{s,t+1}, \quad (66)$$

$$0 \leq s \leq \alpha - 1, 0 \leq t \leq \beta - 1; \quad (67)$$

$$R_{s,\beta} = \lambda_k F_{s,\beta} + (s+1)F_{s+1,\beta}, \quad s = 0, \dots, \alpha; \quad (68)$$

$$R_{\alpha,t} = \lambda_k F_{\alpha,t} + (t+1)F_{\alpha,t+1}, \quad t = 0, \dots, \beta; \quad (69)$$

$$R_{\alpha,\beta} = \lambda_k F_{\alpha,\beta}. \quad (70)$$

By the Diophantine conditions (26), we have that  $\lambda_k \neq 0$ . Consequently, the above linear equations can be recursively solved, starting with the solution  $F_{\alpha,\beta} = \lambda_k^{-1} R_{\alpha,\beta}$ . Indeed, writing  $u^s = (F_{s,0}, \dots, F_{s,\beta})$  and  $v^s = (R_{s,0}, \dots, R_{s,\beta})$ , equation (66) gives

$$v^s = (\lambda_k I_{\beta+1} + A)u^s + (s+1)I_{\beta+1}u^{s+1}, \quad (71)$$

where  $I_{\beta+1}$  denotes the identity matrix of order  $\beta+1$  and the matrix  $A \in gl(\beta+1, \mathbb{C})$  with  $A_{i,i+1} = i$  for  $i = 1, \dots, \beta$  and  $A_{ij} = 0$  elsewhere. By equations (69) and (70) and the Diophantine conditions, the components of  $u^\alpha$  are uniquely determined by  $R$ . Now the recurrence relation (71) gives all  $u^s$  for  $s = \alpha - 1, \dots, 0$ .  $\square$

**Remark A.4.** After replacing  $\mathcal{F}_r$  by the subspace  $\mathcal{R}_r$  of real power series in the above proof, we see that lemma A.3 also holds when  $\text{ad}_r \tilde{S}$  and  $\text{ad}_r \tilde{H}_2$  are restricted to the subspace  $\mathcal{R}_r$ .

## A.2. Non-linear normal form theory

In this subsection, we carry out a normalization procedure for  $H = H_{\omega,\mu}$  as in (57), with respect to the lowest order term  $\tilde{H}_2$  as in (60), up to terms of degree  $r$ . Roughly speaking, such a normalization kills non-integrable terms that appear in the lower order terms in  $\tilde{H}$ , by using time-one flows generated by Hamiltonian functions in  $\text{im ad } \tilde{H}_2$ . We fix the parameters  $(\omega, \mu_1)$ , which satisfy the Diophantine conditions (26). For simplicity, we work in the real-analytic setting; that is, everything is assumed to be real analytic. Under such circumstances, we have the following theorem.

**Theorem A.5 (Quasi-periodic normal form).** *Let  $H = H_v$  be the real-analytic family of integrable Hamiltonians given by (33). Assume that the assumptions of theorem 4.3 are satisfied and that the parameter  $v = (\omega, \mu, \rho) \in \Gamma_{\tau,\gamma}(U')$ , see section 4.1, where  $U$  is a small neighbourhood of the fixed parameter  $v_0$ . Then, for any given even integer  $r \geq 6$ , any fixed  $(\omega, \mu_1)$  satisfying the Diophantine conditions (26) and for any real-analytic Hamiltonian  $\tilde{H}$  sufficiently close to  $H$  in the compact-open topology, there exists a family of symplectic maps*

$$\Phi : \mathbb{T}^m \times \mathbb{R}^m \times \mathbb{R}^4 \times \mathbb{R} \longrightarrow \mathbb{T}^m \times \mathbb{R}^m \times \mathbb{R}^4 \times \mathbb{R},$$

*being real-analytic in  $(x, y, z)$  and  $C^\infty$ -near-the-identity map, such that the Hamiltonian  $\tilde{H} \circ \Phi$  is decomposed into  $G_{\text{int}} + R$ , where the integrable part  $G_{\text{int}}$  takes the form*

$$G_{\text{int}} = \langle \omega, y \rangle + \mu_1 S + N + \mu_2 M + \frac{1}{2} b M^2 + c_1 S M + c_2 S^2 + \sum_{\ell+2|p|+s+t=3}^{r/2} c_{\ell,p,s,t} \mu_2^\ell y^p S^s M^t,$$

*while the non-integrable remainder  $R = R(x, y, z, \mu_2)$  has the Taylor-expansion of the form*

$$R = \sum_{2\ell+4|p|+|q|>r} f_{\ell,p,q}(x) \mu_2^\ell y^p z^q.$$

**Remark A.6.**

- (1) Observe that the integrable parts in this normal form are always of even order in  $z$ , which has to do with the special shape of  $\ker \operatorname{ad}_r \hat{S}$ , see (63).
- (2) If the internal and normal frequencies  $(\omega, \mu_1)$  are considered as parameters, then the normalization map  $\Phi$  as well as the coefficients  $c_{\ell,p,s,l}$  and  $f_{\ell,p,q}$  depend on  $(\omega, \mu_1)$ . By construction of the map  $\Phi = \Phi(x, y, z, \omega, \mu_1, \mu_2)$  as a composition of Hamiltonian time-one flows and the fact that the parameters  $(\omega, \mu_1, \mu_2)$  satisfy the Diophantine conditions (26), one can show, using the ‘standard’ arguments as given in [11, section 2c], that  $\Phi$  depends on  $(\omega, \mu_1, \mu_2)$  in a Whitney-smooth way.

**Proof.** We show the claim by an induction argument. To begin with, we complexify the Hamiltonian function  $\tilde{H}$  by using complex coordinate  $\hat{z} = (\zeta, \bar{\zeta}, \eta, \bar{\eta})$  for  $z = (z_1, z_2, z_3, z_4)$ . In the following, we use the same notation  $\tilde{H}$  for its complexified counterpart. To obtain a real valued normalization map  $\Phi$  and a real valued normal form for  $\tilde{H}$ , we must work in the real subspace  $\mathcal{R}$  instead of in the whole space  $\mathcal{F}$ . For  $r = 2$ , there is nothing to prove, since  $\tilde{H}$  is already in normal form up to degree two. Now suppose that  $\hat{H} \in \mathcal{R}$  is normalized up to (even) degree  $r \geq 2$  and has the form

$$\hat{H} = \sum_{j=1}^{r/2} Z_{2j} + R_{r+1} + R_{r+2} \left( \bmod \prod_{j \geq r+3} \mathcal{R}_j \right),$$

where  $Z_{2j}$  for  $j = 1, \dots, r/2$  are given by

$$Z_{2j} = \sum_{\ell+2|p|+s+t=j} c_{\ell,p,s,t} \mu_2^\ell y^p S^s M^t$$

and where  $R_{r+1} \in \mathcal{R}_{r+1}$  and  $R_{r+2} \in \mathcal{R}_{r+2}$ . Since  $r+1$  is an odd integer, because of the splitting of (63) and (64),  $R_{r+1}$  must belong to  $\operatorname{im} \operatorname{ad}_{r+1} \hat{S}$ . Then, by lemma A.3, there is an  $F_{r+1} \in \operatorname{im} \operatorname{ad}_{r+1} \hat{S}$  such that  $\operatorname{ad}_{\tilde{H}_2}(F_{r+1}) = R_{r+1}$ . Now the time-one map  $e^{\operatorname{ad}_{F_{r+1}}}$  of the Hamiltonian  $F_{r+1}$  transforms<sup>11</sup> the normal form  $\hat{H}$  into

$$\hat{H} \circ e^{\operatorname{ad}_{F_{r+1}}} = \sum_{j=1}^{r/2} Z_{2j} + R_{r+2} \left( \bmod \prod_{j \geq r+3} \mathcal{R}_j \right).$$

This implies that all odd degree terms in  $\hat{H}$  can be removed, compare with remark A.6(1). Next we claim that we can find an  $F_{r+2} \in \operatorname{im} \operatorname{ad}_{r+2} \hat{S}$ , such that

$$\hat{H} \circ e^{\operatorname{ad}_{F_{r+1}}} \circ e^{\operatorname{ad}_{F_{r+2}}} = \sum_{j=1}^{r/2} Z_{2j} + Z_{r+2} \left( \bmod \prod_{j \geq r+3} \mathcal{R}_j \right),$$

where  $Z_{r+2} \in \mathcal{R}_{r+2}$  of the form

$$Z_{r+2} = \sum_{\ell+2|p|+s+t=(r/2)+1} c_{\ell,p,s,t} \mu_2^\ell y^p \tilde{S}^s \tilde{M}^t,$$

where  $\tilde{S} = \zeta \eta - \bar{\zeta} \bar{\eta}$  and  $\tilde{M} = \zeta \bar{\zeta}$ . In other words, we ask for a function  $F_{r+2} \in \mathcal{R}_{r+2}$  such that

$$\operatorname{ad}_{r+2} \tilde{H}_2(F_{r+2}) + Z_{r+2} = R_{r+2}. \quad (72)$$

Write  $F_{r+2} = F^0 + F^1$  and  $R_{r+2} = R^0 + R^1$ , where

$$F^0, R^0 \in \ker \operatorname{ad}_{r+2} \hat{S} \quad \text{and} \quad F^1, R^1 \in \operatorname{im} \operatorname{ad}_{r+2} \hat{S}.$$

<sup>11</sup> In the case  $r = 2$ , the transformation generated by  $F_3 = F_{r+1}$  changes the terms in  $R_4 = R_{r+2}$ .

Again by lemma A.3, the function  $F^1$  can be chosen so that  $\text{ad}_{r+2}\tilde{H}_2(F^1) = R^1$ ; so the equation (72) is reduced to

$$\text{ad}_{r+2}\tilde{H}_2(F_0) + Z_{r+2} = R^0, \quad (73)$$

in the unknowns  $F^0$  and  $Z_{r+2}$  for a given  $R^0$  in the space  $\ker \text{ad}_{r+2}\hat{S}$ . In view of (63), we can restrict equation (73) to the  $\text{ad}\tilde{H}_2$ -invariant subspace  $\mathcal{R}_{\alpha,\alpha}^{0,\ell,p}$  for fixed indices  $(\ell, p, \alpha)$ , where  $\ell + 2|p| + \alpha = (r/2) + 1$ . Let  $\pi : \mathcal{R}_{\alpha,\alpha}^{0,\ell,p} \rightarrow \mathcal{R}_{\alpha,\alpha}^{0,0,0}$  be the projection defined by

$$\pi \left( \sum_{(s,t)=(0,0)}^{(\alpha,\beta)} F_{s,t} \mu_2^\ell y^p \zeta^s \bar{\zeta}^t \eta^{\alpha-s} \bar{\eta}^{\alpha-t} \right) = \sum_{(s,t)=(0,0)}^{(\alpha,\beta)} F_{s,t} \zeta^s \bar{\zeta}^t \eta^{\alpha-s} \bar{\eta}^{\alpha-t}.$$

Observe that  $P_{2\alpha} = \mathcal{R}_{\alpha,\alpha}^{0,0,0}$  is the space of all homogeneous polynomials in  $(\zeta, \bar{\zeta}, \eta, \bar{\eta})$  of degree  $2\alpha$ . Following [51, 61], we have

$$P_{2\alpha} = (\text{im ad } \tilde{H}_2|P_{2\alpha}) \oplus (\ker \text{ad } \tilde{M}|P_{2\alpha}), \quad (74)$$

which induces the direct sum splitting

$$\mathcal{R}_{\alpha,\alpha}^{0,\ell,p} = (\text{im ad } \tilde{H}_2|\mathcal{R}_{\alpha,\alpha}^{0,\ell,p}) \oplus (\ker \text{ad } \tilde{M}|\mathcal{R}_{\alpha,\alpha}^{0,\ell,p}). \quad (75)$$

Moreover,  $\ker \text{ad } \tilde{M}|P_{2\alpha}$  as a real subspace is generated by monomials  $\tilde{S}^s \tilde{M}^t$ , where  $s + t = \alpha$ , see again [51, 61], implying that  $\ker \text{ad}_{r+2}\tilde{M} \subseteq \mathcal{R}_{r+2}$  (as a real subspace) is generated by  $\mu_2^\ell y^p \tilde{S}^s \tilde{M}^t$ , where  $\ell + 2|p| + s + t = (r/2) + 1$ . Decompose  $R^0 = R_0^0 + R_1^0$ , where  $R_0^0 \in \ker \text{ad } \tilde{M}|\mathcal{R}_{\alpha,\alpha}^{0,\ell,p}$  and  $R_1^0 \in \text{im ad } \tilde{H}_2|\mathcal{R}_{\alpha,\alpha}^{0,\ell,p}$ ; there exists an  $F^0$  such that  $\text{ad}_{r+2}\tilde{H}_2(F^0) = R_1^0$ . Such an  $F^0$  and  $Z_{r+2} = R_0^0$  are desired solutions of the linear equation (73). Now the proof is complete by noting that  $Z_{r+2}$  is a quasi-homogeneous polynomial in  $(\mu_2, y, \tilde{S}, \tilde{M})$  of degree  $(r/2) + 1$ , with the weight  $(\alpha_{\mu_2}, \alpha_y, \alpha_{\tilde{S}}, \alpha_{\tilde{M}}) = (1, 2, 1, 1)$ .  $\square$

## References

- [1] Abraham R and Marsden J E 1978 *Foundations of Mechanics* (New York: Benjamin/Cummings) (Advanced Book Program)
- [2] Arnol'd V I 1963 Proof of a theorem of A N Kolmogorov on the preservation of conditionally periodic motions under a small perturbation of the Hamiltonian *Russ. Math. Surv.* **18** 9–36
- [3] Arnol'd V I 1971 Matrices depending on parameters *Russ. Math. Surv.* **26** 29–43
- [4] Arnol'd V I 1989 *Mathematical Methods of Classical Mechanics* (*Graduate Texts in Mathematics* vol 60) (Berlin: Springer)
- [5] Bakhtin V I 1991 Diophantine approximations on images of mappings *Dokl. Akad. Nauk BSSR* **35** 398–400
- [6] Bakhtin V I 1991 Diophantine approximations on images of mappings *Dokl. Akad. Nauk BSSR* **35** 475
- [7] Bates L and Zou M R 1993 Degeneration of Hamiltonian monodromy cycles *Nonlinearity* **6** 313–35
- [8] Birkhoff G D 1966 *Dynamical Systems* (*American Mathematical Society Colloquium Publications* vol IX) (Providence, RI: American Mathematical Society) (addendum by Jürgen Moser)
- [9] Bourgain J 1994 Construction of quasi-periodic solutions for Hamiltonian perturbations of linear equations and applications to nonlinear PDE *Int. Math. Res. Not.* **1994** 475–97
- [10] Bourgain J 1997 On Melnikov's persistency problem *Math. Res. Lett.* **4** 445–58
- [11] Braaksma B L J and Broer H W 1987 On a quasiperiodic Hopf bifurcation *Ann. Inst. H. Poincaré Anal. Non Lin.* **4** 115–68
- [12] Braaksma B L J, Broer H W and Huitema G B 1990 Bifurcations of quasi-periodic tori *Mem. Am. Math. Soc.* **83** no 421, 83–175
- [13] Broer H W 1981 Formal normal form theorems for vector fields and some consequences for bifurcations in the volume preserving case *Dynamical Systems and Turbulence, Warwick 1980* (Coventry, 1979/1980) (*Lecture Notes in Mathematics* vol 898) (Berlin: Springer) pp 54–74
- [14] Broer H W, Cushman R H, Fasso F and Takens F 2007 Geometry of KAM tori for nearly integrable Hamiltonian systems *Ergod. Theory Dyn. Syst.* at press

- [14] Broer H W, Hanßmann H, Jorba À, Villanueva J and Wagener F O O 2003 Normal-internal resonances in quasi-periodically forced oscillators: a conservative approach *Nonlinearity* **16** 1751–91
- [15] Broer H W, Hanßmann H and You J 2005 Bifurcations of normally parabolic tori in Hamiltonian systems *Nonlinearity* **18** 1735–69
- [16] Broer H W, Hoo J and Naudot V 2007 Normal linear stability of quasi-periodic tori *J. Diff. Eqns* **232** 355–418
- [17] Broer H W, Hoveijn I and van Noort M 1998 A reversible bifurcation analysis of the inverted pendulum *Physica D* **112** 50–63
- [18] Broer H W, Hoveijn I, van Noort M and Vegter G 1999 The inverted pendulum: a singularity theory approach *J. Diff. Eqns* **157** 120–49
- [19] Broer H W, Huitema G B and Sevryuk M B 1996 *Quasi-Periodic Motions in Families of Dynamical Systems (Lecture Notes in Mathematics vol 1645)* (Berlin: Springer)
- [20] Broer H W, Huitema G B and Takens F 1990 Unfoldings of quasi-periodic tori *Mem. Am. Math. Soc.* **83** no 421, 1–82
- [21] Broer H W and Sevryuk M B 2007 KAM theory: quasi-periodicity in dynamical systems *Handbook of Dynamical Systems* vol 3 ed H W Broer *et al* (Amsterdam: North-Holland)
- [22] Broer H W and Vegter G 1992 Bifurcational aspects of parametric resonance *Dynamics Reported New Series 1* (Berlin: Springer) pp 1–53
- [23] Chierchia L and Gallavotti G 1982 Smooth prime integrals for quasi-integrable Hamiltonian systems *Nuovo Cimento B* **67** 277–95
- [24] Cushman R H and Bates L M 1997 *Global Aspects of Classical Integrable Systems* (Basel: Birkhäuser)
- [25] Cushman R H and Duistermaat J J 1988 The quantum mechanical spherical pendulum *Bull. Am. Math. Soc. (N.S.)* **19** 475–9
- [26] Cushman R H, Dullin H R, Giacobbe A, Holm D D, Joyeux P, Lynch M, Sadovskii D A and Zhilinskii B I 2004 CO<sub>2</sub> molecule as a quantum realization of the 1 : 1 : 2 resonant swing-spring with monodromy *Phys. Rev. Lett.* **93** 024302(4)
- [27] Cushman R H and Knörrer H 1985 The energy momentum mapping of the Lagrange top *Differential Geometric Methods in Mathematical Physics (Clausthal, 1983) (Lecture Notes in Mathematics vol 1139)* (Berlin: Springer) pp 12–24
- [28] Cushman R H and van der Meer J-C 1990 The Hamiltonian Hopf bifurcation in the Lagrange top *Géométrie Symplectique et Mécanique (La Grande Motte, 1988) (Lecture Notes in Mathematics vol 1416)* (Berlin: Springer) pp 26–38
- [29] Cushman R H and Sadovskii D A 2000 Monodromy in the hydrogen atom in crossed fields *Physica D* **142** 166–96
- [30] Duistermaat J J 1980 On global action-angle coordinates *Commun. Pure Appl. Math.* **33** 687–706
- [31] Dullin H R and Vũ Ngọc San 2004 Vanishing twist near focus-focus points *Nonlinearity* **17** 1777–85
- [32] Efstathiou K 2005 *Metamorphoses of Hamiltonian Systems with Symmetries (Lecture Notes in Mathematics vol 1864)* (Berlin: Springer)
- [33] Efstathiou K, Cushman R H and Sadovskii D A 2004 Hamiltonian Hopf bifurcation of the hydrogen atom in crossed fields *Physica D* **194** 250–74
- [34] Eliasson L H 1988 Perturbations of stable invariant tori for Hamiltonian systems *Ann. Sc. Norm. Sup. Pisa, Cl. Sci., Ser. IV* **15** 115–47
- [35] Eliasson L H 1990 Normal forms for Hamiltonian systems with Poisson commuting integrals—elliptic case *Comm. Math. Helv.* **65** 4–35
- [36] Gibson C G 1979 *Singular Points of Smooth Mappings (Research Notes in Mathematics vol 25)* (Boston, MA: Pitman) (Advanced Publishing Program)
- [37] Goldstein H 1950 *Classical Mechanics* (Reading, MA: Addison-Wesley)
- [38] Graff S M 1974 On the Conservation of hyperbolic invariant tori for Hamiltonian systems *J. Diff. Eqns* **15** 1–69
- [39] Gustavson F G 1966 On constructing formal integrals of a Hamiltonian system near an equilibrium point *Astron. J.* **71** 670–86
- [40] Hanßmann H 1998 The quasi-periodic centre-saddle bifurcation *J. Diff. Eqns* **142** 305–70
- [41] Hanßmann H 2007 *Local and Semi-Local Bifurcations in Hamiltonian Dynamical Systems—Results and Examples (Lecture Notes in Mathematics vol 1893)* (Berlin: Springer)
- [42] Hanßmann H and van der Meer J-C 2005 On non-degenerate Hamiltonian Hopf bifurcations in 3DOF systems *Equadiff 2003, Hasselt 2003* ed F Dumortier *et al* (Singapore: World Scientific) pp 476–81
- [43] Hausdorff F 1962 *Set theory* 2nd edn (New York: Chelsea) (translated from the German by J R Aumann *et al*)
- [44] Huitema G B 1988 Unfoldings of quasi-periodic tori *PhD Thesis* University of Groningen
- [45] de Jong H H 1999 Quasiperiodic breathers in systems of weakly coupled pendulums *PhD Thesis* Rijksuniversiteit Groningen

- [46] Kolmogorov A N 1954 On conservation of conditionally periodic motions for a small change in Hamilton's function *Dokl. Akad. Nauk SSSR (N.S.)* **98** 527–30
- [47] Kuksin S B 1989 Perturbation theory of conditionally periodic solutions of infinite-dimensional Hamiltonian systems and its applications to the Korteweg-de Vries equation *Math. USSR-Sb.* **64** 397–413
- [48] Landau L D and Lifshitz E M 1960 *Mechanics (Course of Theoretical Physics)* vol 1 (Oxford: Pergamon) (translated from the Russian by J B Bell)
- [49] Li Y and Yi Y 2005 Persistence of lower dimensional tori of general types in Hamiltonian systems *Trans. AMS* **357** 1565–600
- [50] Marsden J E and Scheurle J 1993 Lagrangian reduction and the double spherical pendulum *Z. Angew. Math. Phys.* **44** 17–43
- [51] van der Meer J-C 1985 *The Hamiltonian Hopf Bifurcation (Lecture Notes in Mathematics* vol 1160) (Berlin: Springer)
- [52] van der Meer J-C 1990 Hamiltonian Hopf bifurcation with symmetry *Nonlinearity* **3** 1041–56
- [53] Mel'nikov V K 1965 On certain cases of conservation of almost periodic motions with a small change of the Hamiltonian function *Dokl. Akad. Nauk SSSR* **165** 1245–8
- [54] Mel'nikov V K 1968 A certain family of conditionally periodic solutions of a Hamiltonian system *Dokl. Akad. Nauk SSSR* **181** 546–9
- [55] Meyer K R and Hall G R 1992 *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem (Applied Mathematical Sciences* vol 90) (Berlin: Springer)
- [56] Moser J K 1967 Convergent series expansions for quasi-periodic motions *Math. Ann.* **169** 136–76
- [57] Nguyen Tien Zung 1996 Kolmogorov condition for integrable systems with focus–focus singularities *Phys. Lett. A* **215** 40–4
- [58] Nguyen Tien Zung 1997 A note on focus–focus singularities *Diff. Geom. Appl.* **7** 123–30
- [59] van Noort M 2001 Global coherent dynamics of the parametrically forced pendulum: a case study in one-a-half degree of freedom *PhD Thesis* University of Groningen
- [60] Ollé M, Pacha J R and Villanueva J 2005 Dynamics close to a non semi-simple 1 : –1 resonance periodic orbit *Discrete. Contin. Dyn. Syst. B* **5** 799–816
- [61] Pacha J R 2002 On the quasi-periodic Hamiltonian Andronov–Hopf bifurcation *PhD Thesis* Universitat Politècnica de Catalunya
- [62] Pöschel J 1982 Integrability of Hamiltonian systems on Cantor sets *Commun. Pure Appl. Math.* **35** 653–96
- [63] Pöschel J 1989 On elliptic lower-dimensional tori in Hamiltonian systems *Math. Z.* **202** 559–608
- [64] Poston T and Stewart I 1978 *Catastrophe Theory and its Applications (Surveys and Reference Works in Mathematics, No 2)* (Boston, MA: Pitman) (with an appendix by D R Olsen, S R Carter and A Rockwood)
- [65] Rajiu T and van Moerbeke P 1982 The Lagrange rigid body motion *Ann. Inst. Fourier (Grenoble)* **32** 211–34
- [66] Rink B W 2004 A Cantor set of tori with monodromy near a focus-focus singularity *Nonlinearity* **17** 347–56
- [67] Rüssmann H 2001 Invariant tori in non-degenerate nearly integrable Hamiltonian systems *Regul. Chaotic Dyn.* **6** 119–204
- [68] Sevryuk M B 2007 Invariant tori in quasi-periodic non-autonomous dynamical systems via Herman's method *Discrete Contin. Dyn. Syst. A*, at press
- [69] Stewart I 2004 Quantizing the classical cat *Nature* **430** 731–2
- [70] Stoker J J 1950 *Nonlinear Vibrations in Mechanical and Electrical Systems* (New York: Interscience)
- [71] Vũ Ngọc San 2003 On semi-global invariants for focus-focus singularities *Topology* **42** 365–80
- [72] Whittaker E T 1944 *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (New York: Dover)
- [73] Williamson J 1936 On the algebraic problem concerning the normal form of linear dynamical systems *Am. J. Math.* **58** 141–63
- [74] Xu J and You J 2001 Persistence of lower-dimensional tori under the first Melnikov's non-resonance condition *J. Math. Pures Appl.* **80** 1045–67
- [75] You J 1999 Perturbations of lower-dimensional tori for Hamiltonian systems *J. Diff. Eqns* **152** 1–29